Explanatory Indispensability and the Set Theoretic Multiverse

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Abstract

Width multiverse approaches to set theory (like Joel David Hamkins' influential proposal in [6]) reject the idea that there's an intended width hierarchy of sets which contains 'all possible subsets' of the sets that it contains. In this paper, I raise an explanatory indispensability worry for the multiverse theorist and distinguish three different possible styles of response to this worry. I will argue that each approach faces some serious prima facie problems. And I'll suggest that, by clarifying their response to this puzzle about applications, multiverse theorists can helpfully clarify their proposals concerning pure mathematics.

1 Introduction

On a conventional understanding of set theory, there's a unique intended hierarchy of sets that contains, at each layer, sets corresponding to 'all possible ways of choosing' sets from lower levels. This determines a unique intended right answer to all set-theoretic questions, like the continuum hypothesis (CH), whose truth value only depends on the width of the hierarchy of sets.

In contrast, what I will call width multiverse theories (influentially exemplified by Hamkins in [6]) agree that there are Platonic mathematical objects, the sets, but deny that there's a unique intended hierarchy of sets (even up to width). Instead, they take there to be a multiverse of different hierarchies of sets (set theoretic universes). And they maintain that there's no general intended right answer to certain set-theoretic questions whose truth value varies between universes. Rather mathematicians simply choose, in different contexts, to work in one kind of universe or another. In particular, the width multiverse theorists I'll be concerned with in this paper (henceforth, I will just call them 'multiverse theorists') accept the following claims.

- Sets exist
- For every set-theoretic universe V, there is a strictly wider universe V[G] corresponding to (what is called) a forcing extension of V. This V[G] contains all the sets in the original universe but adds extra subsets of sets in the original universe.
- There's no context-independent right answer to questions like CH (the continuum hypothesis) whose truth value varies between set theoretic universes [1].

In this paper, I will discuss a challenge for multiverse theorists generally, and especially for Hamkins – who advocates a particularly bold form of multiverse theory which also denies there's a unique intended natural number structure among other things.

Hamkins' multiverse theory suggests that we should accept a wide range

[1]Technically, one could satisfy these three requirements by adopting a form of width multiversism which only claims the multiverse is closed under *certain kinds* of forcing extensions for each universe V, chosen so as to leave the truthvalue of CH (or certain other sentences) fixed.

In what follows, I will restrict my attention to forms of width multiversism which take the truthvalue of CH to vary between universes, as most popular forms of width multiversism do. For almost all actual multiverse theorists in the literature deny that there's a favored right answer to CH. And indeed I take a common motivation for interest in multiverse theories to be the intuition that there's no right answer to CH.

However, the main worry proposed in §2.1 arises for all width multiverse theorists (as defined above). And most of my arguments will also apply to all such views.

of universes as equally real (and having equal intrinsic intendedness). He asserts that each set theoretic universe has a forcing extension in the multiverse, alongside the following more radical closure principles:

- Every universe looks countable from the point of view of some larger universe (Countability Principle).
- Every universe's copy of the ordinals looks ill-founded from the point of view of some larger universe (Well-Foundedness Mirage).

Notably, the latter principle has the result that there's no intended model of the numbers/fixed concept of natural numbers which picks out a unique structure. So Hamkins holds that the truth value of even claims about the numbers can change between universes (something which mere forcing extensions cannot alter, and which other width multiverse theorists need not accept).

Crudely, the explanatory challenge I have in mind goes like this. When giving scientific explanations, we currently are open to scientific hypotheses (conventionally formalized by ontologists using sets) which invoke facts about 'all possible ways of choosing'[2]. But accepting width multiverse theory seems to require rejecting this notion (or at least denying traditional claims about how it connects to set theory). Thus, accepting width multiverse theory threatens to commit one to ruling out seemingly cogent candidate explanations for physical facts, a priori. I will develop the above worry and consider various ways Hamkins and other width-multiverse theorists could try to answer it.

In §2 I'll clarify what I mean by a lazy explanatory indispensability argument and develop a particular lazy explanatory indispensability argument against multiverse theorists. In the remaining sections, I'll discuss three styles of response to this worry. In §3 and §5 I'll assess a pair of strategies which deny

[2]Cf. arguments in thermodynamics that appeal to facts about sets coding all possible configurations of particles compatible with some macroscopic description to explain why entropy increases. Thanks to REDACTED for suggesting this example.

that there's a favored notion of 'all possible ways of choosing'. In §4 I'll assess responses which take facts about all possible ways of choosing to be reflected by facts about what sets exist within the multiverse as a whole. I will argue that each approach faces some important difficulties.

2 The Explanatory Indispensability Worry

So, let's begin with the very idea of a lazy explanatory indispensability argument. Classic explanatory indispensability arguments [15, 13, 1, 4] against mathematical nominalism maintain that we should accept mathematical objects, because the physical theories which best predict and explain certain empirical data can't be formulated without quantifying over them. But in this paper I'll present a slightly different challenge, which differs from the above classic explanatory indispensability arguments in two ways.

First, my argument attacks mathematical truth value antirealism[3], not mathematical object anti-realism. It doesn't argue that *mathematical objects* are needed to give certain kinds of intuitively good scientific explanations. After all, width multiverse theorists already accept the existence of plenty of sets![4] Rather it

[3]Here, by truthvalue realism, I mean truthvalue determinacy (i.e. taking there to be right answers to various mathematical questions regardless of whether our proof practices let us decide these question). A width multiverse theorist like Hamkins could argue that (in a very different sense) they are more realist than a traditional Platonist. For they could say that by positing more universes they are positing 'more truths' —in the sense of more truth patterns. Rather than merely positing truths about one universe they are positing truths about a range of different mathematical universes).

[4]I take Hamkins' multiverse view to be, as he says, a form of Platonism[7], which combines ontological realism (the view that mathematical objects like sets exist) with significant truth value anti-realism (the view that that there isn't a single right answer to many questions in the language of set theory). For example, Hamkins writes, "In this article, I shall argue for... the multiverse view, which holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist."[7]. I take this to commit him to the existence of the sets within various set theoretic universes (since traditional single universe theorists would say that sets within a favored set universe exist as platonic objects). However dialectically this doesn't matter. I'm just clarifying that my lazy explanatory indispensability argument will target Hamkins' anti-realism, rather than criticising his rejection of mathematical objects (as classic Quine-Putnam explanatory indispensability arguments do).

argues for accepting more *truth-value realism*/determinacy, by accepting that we can latch onto a unique intended width for the hierarchy of sets which pins down proof transcendent right answers to questions like CH.

Second, my argument is (what I'll call) a *lazy* explanatory indispensability argument in the following sense. It poses an a priori rather than a posteriori challenge. Unlike classic explanatory indispensability arguments, it doesn't try to point out a part of our actual best scientific theory that advocates of the view being criticized (in this case multiverse theory) can't adequately express, or data they can't adequately explain. Instead, it suggests that multiverse theory implausibly rules out certain seemingly-cogent physical hypotheses a priori.

To explain this, consider Baker's classic explanatory indispensability argument, that nominalists can't match the goodness of Platonist explanations for prime length life cycles of cicadas. If you learned Baker's evidence for many cicadas species having prime length life cycles was a hoax, would this fully quash Baker's challenge to nominalism? Not necessarily. For even if we don't need such an explanation to account for current experience, many would be hesitant to accept any philosophy of mathematics that required us to stop considering such explanations as a live option.

In the next subsection, I'll argue that Hamkins and other multiverse theorists face an analogous worry. Accepting width multiverse theory threatens to imply implausible restrictions on the space of candidate physical explanatory hypotheses.

Specifically, I will note that there are (intuitively) metaphysically possible situations in which certain physical regularities are explained by appeal to facts about logical possibility/all possible ways of choosing. And I will argue that multiverse theorists face prima facie difficulties accounting for this.

2.1 Core Worry

So now let's turn to developing the specific lazy explanatory indispensability argument I want to press.

From a traditional point of view, we seem to have a modal notion of 'all possible ways of choosing', which has close a priori connections to both set theory (on the other) and counterfactual-supporting constraints on non-mathematical reality (on the one hand) as follows.

- Facts about 'all possible ways of choosing' are supposed to help determine a unique intended structure for the set-theoretic universe (up to width). For each layer of the iterative hierarchy of sets is supposed to contain sets corresponding to all possible ways of choosing some sets first occuring at lower levels.

- Facts about 'all possible ways of choosing' are supposed to reflect counterfactual supporting constraints on constrain non-mathematical reality, in a way that can help predict and explain regularities involving physical objects.

Suppose we have a (finite or infinite) physical map[5] which has never been three-colored, despite many changes in the colors of individual map regions. That is, suppose there's never been a point at which each map region is either red, green or blue but no two adjacent map regions have the same color

This fact can be stated using only first order logic and non-mathematical vocabulary like 'is a map tile', 'is adjacent to'. We don't need quantify over objects, or employ second order quantifiers or any distinctively mathematical

[5]Presumably, there aren't really any infinite physical maps. Perhaps one could make the cases more realistic by appealing to infinitely many galaxies (or points in physical space or point particles) with adjacency relations between them. However, I won't attempt to do that here. Instead, I'll merely appeal to the apparent conceivability of certain scenarios involving mathematically explained regularities concerning physical maps, to give a lazy explanatory indispensability argument in the sense described above.

vocabulary. So presumably it's something people with different approaches to set theory can agree on. [6]

(From a traditional point of view) it seems that a true and illuminating explanation for the fact that our map *has never been* three colored could be that the map in question is not three color*able*, in a modal sense (reflected by the fact that there's no set coding a three coloring function in the hierarchy of sets with ur-elements).

> **Infinite Map Non-Three Coloring Explanation** (traditional universe-ist version): The map isn't 3-colored because there is no set coding a 3-coloring function and, since every possible way of choosing is witnessed by a set, if there were a way of 3-coloring the map there would be such a set.

If this explanation is correct, we think three things follow. First, the map isn't actually three-colored. Second, the map couldn't 'easily' have been three-colored (i.e., it isn't three-colored at any close possible worlds)[7]. Third, what rules out the map actually being three-colored is a general logical/combinatorial constraint (i.e., one that applies analogously to all predicates and relations) which also implies the map isn't (and couldn't easily have been) three-scented or three-textured either.

But what happens to this picture if we adopt a multiverse understanding of set theory? As noted above, the multiverse theorist must deny that there's a 'full width' hierarchy of sets which contains all possible subsets of sets it

[6] Formalizing the claim that the map isn't (currently) three colored in first order logic is straightforward. How you formalize the claim that it has never *been* three colored (in a certain interval of time) may depend on details of your favored logic for dealing with events/temporal logic. But no appeal to numbers or sets seems required, if you are allowed to quantify over times, and pick out endpoints of the period during which we're claiming the map was not three colored via definite descriptions that don't involve numbers.

[7] Plausibly (in typical cases) all very close metaphysically possible worlds preserve the way that map regions are related by adjacency, and relevant logico-combinatioral constraints apply with metaphysical necessity. So the map won't be three colored at these close possible worlds either.

contains[8]. So it seems they must either

- Reject the above notion of 'all possible ways of choosing' or
- Accept this notion but say that (for some reason) no single hierarchy of sets can contain all possible subsets of sets it contains.

Thus, the multiverse theorist faces a question about seemingly cogent physical explanatory hypotheses which appeal to this notion of all possible ways of choosing (via set theory). It's hard to deny that something like (some version) these claims express a legitimate explanatory hypothesis. But it is not clear how to make such physical explanations compatible with multiverse set theory.

Thus, we get a lazy explanatory indispensability argument.

Let me note two things about this challenge.

First, the worry at issue doesn't just arise when different universes in the multiverse are supposed to *disagree* on the truth-value of mathematical claims relevant to some a physical explanatory hypothesis[9]. Rather it applies to all explanations which assume a connection between set theory and law-like constraints on physical objects. As we have seen, the multiverse theorist rejects the traditional bridge between set theory and non-mathematical reality (the assumption that facts about sets reflect general law-like constraints on 'all possible ways of choosing' which constrain how any relations can apply to physical objects). But once we demolish this bridge, it becomes prima facie unclear why the non-existence of certain sets (e.g. sets coding a three coloring) — in one universe, or the multiverse as a whole — should imply *anything* about how physical properties can apply to physical objects.

[8]For, they think each universe V exists alongside an expanded universe V[G] which adds 'missing subsets' of some sets that are already in V.

[9]In fact, as we will see in §5, there are special reasons the multiverse theorist should say all universes agree that there's no three coloring function, in cases where a map intuitively isn't three colorable. But these reasons don't apply to other seemingly cogent physical explanatory hypotheses like the troop distribution example below.

Second, the physical explanatory hypotheses we need to account for can have different logical forms and properties. For example, we might say the three coloring explanation invokes a kind of $\neg\exists$ claim (that there's no set with a certain property). But other seemingly cogent explanatory hypotheses have a more complex structure ($\forall\exists$ rather than $\neg\exists$), like the following.

> **Troop Distribution**: The reason why no one has succeeded in holding such-and-such map region is that, for every possible way of stationing defending troops in countries on the map satisfying ... constraints, there's a way of stationing attacking troops such that ...[10]

With this in mind, I will discuss three possible strategies a multiverse theorist could use to respond to the above lazy indispensability challenge.

3 A Physically Preferred V_p

The first approach I want to consider accepts that there is no unique favored notion of "all possible ways of choosing". It replaces traditional scientific-explanatory appeals to a unique favored hierarchy of sets with ur-elements containing sets witnessing 'all possible ways of choosing' some physical objects, with corresponding claims about a certain specified universe V_p in the multiverse. This universe V_p is claimed to be *physically special*, in containing sets witnessing lawlike constraints on how it would be physically possible for any *sufficiently physically definable* relations to relate physical objects.[11]

[10]Presumably, there aren't really any infinite physical maps. Perhaps one could make the above example more realistic by appealing to infinitely many galaxies (or points in physical space or point particles) with adjacency relations between them. However, I won't attempt to do that here. Instead, I'll merely appeal to the apparent conceivability of certain scenarios involving mathematically explained regularities concerning physical maps, to give a lazy explanatory indispensability argument in the sense described above.

[11]This strategy can be somewhat motivated by Hamkins' remark that there are surprisingly deep analogies between his favored approach to set theory and (a certain version of) common

More specifically, the idea is that physical law prevents either initial conditions or, say, the results of physically random events from ever letting physical properties apply in a way that isn't (already) witnessed by the existence of a corresponding set in V_p.[12] So, for example, suppose that the world contains some infinite sequence of different objectively physically random independent coinflips (or spin measurements). On this proposal, we'd say that physical law prevents the set of coins that come up heads from being a set that occurs in some other universe V^* but not V_p. We'd say that (as a matter of physical law) V_p currently/actually contains sets corresponding to both the actual extension of

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contemporary pluralism about geometry[6]. Arguably physical and mathematical discoveries in the early 20th century support separating physical from mathematical geometry as follows. Many geometrical axioms can be legitimately studied within pure mathematics. However, there's a physically correct geometry – one that comes out true on physically intended interpretations of 'point' and 'line', and thereby reflects the true structure of physical space and physically necessary counterfactual-supporting constraints on the behavior of physical objects.

Switching to this pluralist/truthvalue anti-realist approach to the geometry doesn't generate an explanatory indispensability problem. For we can simply replace appeals to facts about one true geometry (implying counterfactual supporting constrains on physical objects' spatial relations) with appeals to this physically favored geometry, without loss of explanatory or unifying power. So, for example, we can appeal to facts about the physically favored geometry (aka facts about the structure of physical space) when explaining why round manhole covers are useful or why we should expect to get certain results when measuring land.

Accordingly, a multiverse theorist who is inspired to treat set theory and geometry analogously might might answer my lazy explanatory indispensability challenge in the way proposed in this section.

[12]Specifically, the multiverse theorist might say that it's a **physical law** that physical properties can't apply to physical objects in a way that would let some formula $\phi(x, o_1, ..o_n)$ (with only physical properties and logical relations in ϕ and only physical objects $o_1 \ldots o_n$ – or perhaps sets in V_p – as parameters) pick out a set of physical objects that isn't already in V_p, as follows.

- **Physical Separation**: if x is a set in V_p, then V_p also includes all 'sufficiently physically definable' subsets of x. So, for example, it satisfies Separation for all natural languge expressions ϕ with parameters ranging over physical objects and sets in V_p, but not other sets in other universes in the multiverse. Here are some examples of what this principle requires.
  - Since (by the ur-element principle U) V_p contains a set of all physical objects, this principle tells us V_p must contain a set of red physical objects (and green ones, positively charged ones etc.).
  - If V_p contains a pure set x (say, its version of the numbers) and a function f from the numbers to the marbles (e.g. f(n) might be the n-th marble you've seen in your life), then this principle tells us that V_p must also contain a set of the elements of x such that f(x) is a red marble.
- The above claims holds with physical necessity. That is, the laws of physics prevent any physically definable property from picking out a missing subset of V_p (i.e., applying to some physical objects in a way that is not *already* 'witnessed' by a set that actually exists in V_p).

'is a coin that came up heads' and all *physically possible* extensions this predicate could have had. [13].

The multiverse theorist could then reformulate our sample three-colorability explanation to say the following (and treat other physical explanations invoking the notion of all possible ways of choosing via set theory analogously).

> There's a certain physically preferred set theoretic universe V_p within the multiverse, which reflects lawlike constraints on how *all physically definable properties* can apply to actually existing objects, in the following sense. For all existing objects xx and physically definable property ϕ, it would be physically impossible for ϕ to apply to some yy among the xx, without V_p already (actually) containing a set with exactly these objects yy its elements.
>
> There is no set witnessing a way of three-coloring a map in this physically preferred V_p.
>
> Therefore the map isn't three-colored – and, indeed, it would be physically impossible for it to be three-colored (while facts about how map tiles are related by adjacency are held fixed)

Note that the restriction of the above-hypothesized law about V_p to properties which are *physically definable* is not optional. The multiverse theorist might

[13] Alternately (in the same spirit) one could let context do more work, and tell a story along the following lines. The meaning of references to 'all possible ways of choosing' in physical explanations, will be determined by context (which universe we are currently working in), just as the multiverse theorist would say the meaning of talk about "continuum-many objects" will depend on the status of CH in whatever universe the speaker is currently working in.

One could then try to answer my explanatory indispensability challenge by saying that scientists can replace traditional explanations (which appeal to a favored notion of all possible ways of choosing that is supposed to reflect counterfactual-supporting constraints on how any properties could apply) with something equally explanatorily good but multiverse-friendly, as follows. First, get into a context where you are working with a/the physically favored hierarchy of sets with ur-elements. Then utter exactly the same sentences a traditional single universe theorist would have asserted. However, I take this variant proposal to face all the same problems (about giving counterfactual supporting explanations for physical regularities traditionally explained by appeal to all possible ways of choosing) for the main proposal in this section noted below.

have wanted to mirror conventional set theory better by proposing a physical law that V_p contains sets of physical objects corresponding to the extension of every property definable with parameters[14]. However, they can't say this. For, the multiverse view takes V_p (like every universe) to have a universe corresponding to a forcing extension $V_p[G]$ which adds missing subset G of the natural numbers in V_p. But if there is an infinite collection of physical objects xx[15], we can use G as a parameter to define plurality yy from among these physical objects xx, such that V_p doesn't contain a set corresponding to the yy.[16]

This approach has some attractions. For example it lets the multiverse theorist preserve the intuitive counterfactual-supporting force of traditional explanations. And applying it is straightforward.

However, if we take this approach, the question, 'how does physics control the outcomes of seemingly random independent events (e.g., coin tosses), to avoid realizing (i.e., letting us use physical vocabulary to define) a missing subset?' can be troublesome. Maybe it's just a brute physical law that the outcomes of coinflips and painting countries etc. always avoid letting one define the missing subset. But accepting such a law is prima facie uncomfortable.

[14]From a naive/traditional point of view, facts about set theory constrains the physical world because V contains 'all possible subsets', in a way that ensures for each set z in V, V includes all subsets of z which can be defined in the following ways:

- using **any** objects as parameters (not just sets in V) and any relations in our language (not just $\in$)
- using any variant language we might speak in contexts where we add new predicates or names **or drop quantifier restrictions**[12]

So we accept the following schema as holding with metaphysical necessity:

Full Separation Schema (I slightly abuse notation, for legibility, in writing $z \in V$) for V $\Box(\forall z \in V)(\forall w_1)\ldots(\forall w_n)(\exists y \in V)(\forall x)[x \in y \Leftrightarrow ((x \in z) \wedge \phi)]$

The width multiverse theorist might want to mirror this claim, but say that it's physically necessary that V_p has the properties ascribed to V in the Full Separation Schema. However (for reasons discussed in a footnote below), it turns out one cannot.

[15]By this I mean, 'if V_p contains a function f mapping its copy of the natural numbers to these physical objects in a 1-1 way'.

[16]By using the relevant function f and this generic G as parameters, we can define a property (being in the image of G, under f) whose extension cannot be in V_p. For, by the assumption that V_p satisfies basic axioms of set theory with ur-elements like ZFC, if it contained a 1-1 function of f and the image of G under f, it would have to contain G. Positing a physical law which only constrains how *sufficiently physically definable* properties apply lets us get around this problem.

For arguably the concept of physically definable properties is too unnatural to figure in a plausible fundamental physical law.[17][18]

4 Appeal to the Whole Multiverse

Now let's turn to a different response to the lazy explanatory indispensability worry above.

The multiverse theorist might allow that there are genuine (and fully determinate) facts about 'all possible ways of choosing', which constrain how any properties can apply to physical objects, but deny that any single set-theoretic universe can witness all possible ways of choosing as traditionally expected[19]. Rather, (they will say) *the multiverse as a whole* contains sets witnessing 'all possible ways of choosing' some physical objects[20].

Accordingly, we can rewrite traditional physical explanations to replace

[17] Compare this to objections to theories that observation collapses the wave function, on the grounds that observation is the wrong kind of concept to figure in a fundamental physical law.

[18] One might try to avoid the problems above by simply stipulating that, in applied mathematical contexts, we always mean to talk about a set-theoretic universe that happens to satisfy the Physical Separation principle above (i.e., a V_p that contains all subsets of sets it contains that are physically definable, given how physical properties *happen to actually apply*). But such an approach wouldn't offer any genuine explanations, only dormative virtue non-explanation, saying that the map is not three-colored because there is no time t at which the map is three colored.

If we took this approach, the fact that our contextually relevant V_p does not contain any set witnessing a three coloring would indeed *imply* that the map wasn't three colored. But it would not explain the latter fact. For citing this deduction as an explanation for why the set never actually got three colored would be like saying, 'The reason why Jake doesn't have a driver's license is that the list of all people who hold a driver's license doesn't include Jake'.

In addition to being intuitively unacceptable, neither of these would-be explanations preserves the intuitive counterfactual supporting force of our original explanation. They don't rule out the possibility that their explanandum is a complete fluke: that that map could very easily (at very close possible worlds) have been three colored or Jake could very easily have learned to drive.

[19] That is, there is and can be no single universe which contains at each layer sets corresponding to all possible ways of choosing some sets at lower layers.

[20] Note that the strategy I'm currently considering need not assume that there are sets corresponding to all possible ways of choosing from arbitrary *sets* (or sets below some layer, if talk of layers can be legitimately applied) in the multiverse. Rather it (prima facie) only requires assuming that the multiverse contains sets corresponding to something like all possible ways of choosing

- some *physical objects*, as needed to paraphrase claims about how it would be logically possible for properties to apply to physical objects.
- some n-tuples of physical objects, as needed to paraphrase claims about how it would be logically possible for n-place relations to apply to physical objects

claims about the intended set-theoretic universe V with claims that quantify over all sets in all universes in the multiverse[21]. For example, the non-three coloring explanation above can be rewritten as follows.

> No universe anywhere in the multiverse contains a set three coloring the map — and the multiverse contains sets witnessing all possible ways of choosing. Thus, the map isn't three-colored.

I think this response immediately conflicts with the spirit of many width multiverse theories, especially Hamkins' self-admittedly radical proposal[22]. However, it also faces a more concrete problem. I'll argue that assumptions needed for this proposal would let us talk about a unique intended natural number structure (contra Hamkins' rejection of such a structure). And a similar technique can recover an intuitively intended truth value for CH – contra general width multiverse theorists' anti-realism about CH.

The structure of my argument in this section will go as follows. First, I'll argue that if certain infinite collections of physical objects existed, then we could use the ability to quantify over all universes in the multiverse to identify a physical copy of the intuitively intended natural number structure. Then I'll argue that even if such infinite physical structures don't exist in the actual world, we can use talk of what would have to be true in possible worlds where they do exist to cash out intended truth values for claims about the natural numbers.

So, to start, temporarily assume that the Roman emperors under 'ruled after' happen to satisfy PA^i (i.e., the finitely many Peano Axioms for arithmetic, sans

[21] Hamkins does not provide a language for doing this (unsurprisingly, if I'm right that this would be counter to the spirit of his project). But perhaps a multiverse theorist could try to implement this strategy (of allowing facts about what sets exist anywhere in any universe) by including general set and element concepts 'set (in some universe-with-ur-elements)' , 'is element of' and a perhaps a relation '...is a set in universe..' in our language (though the latter move would require introducing universes qua objects, alongside the sets in them).

[22] For example, note that Hamkins doesn't provide formal machinery to quantify over all sets in all universes.

the induction schema, stated in terms of successor)[23]. The response currently being considered assumes that, for any possible way of choosing from the physical objects in the actual world, there is a universe in the multiverse which contains a set S consisting of exactly those physical objects. Given this further assumption, we can define an initial segment of the emperors that (intuitively) forms an ω sequence, as follows.

> x is a **good emperor** iff x is a Roman emperor and x belongs to every set S in some universe in the multiverse which contains the first emperor and is successor closed (i.e., such that for all roman emperors y, if $y \in S$ then the successor of y is also in s).

So (given our assumption that the emperors satisfy PA^{-}), the good emperors must form an ω sequence. For, they will satisfy a version of the second ordered induction principle (being as few as possible while being closed under successor) cashed out in terms of all possible ways of choosing. And, together PA^{i} and this induction principle suffices to uniquely pick out an intended natural number structure.

Thus we will have some physical objects which we intuitively accept as forming a genuine ω sequence (when considered under the relation 'ruled after') – regardless of whether any specially favored model of PA exists among the pure sets. And so arbitrary claims about number theory will be true iff the corresponding claims about good emperors are true. I take this conclusion to be incompatible with Hamkins view that there's no unique intended natural number structure.

Now what about the fact that (in reality) the emperors don't actually[24] satisfy PA^{i}?

[23] The argument I'm making here draws inspiration from [3].

[24] c.f. Russell's troubles formalizing pure mathematics without an axiom of infinity that uncomfortably required the existence existence of infinitely many non-set objects[10].

We can drop this assumption from our argument (while still preserving our intuitive problem for the Hamkins-style multiverse theorist) if we make certain plausible assumptions about metaphysical possibility, listed below. For these assumptions suggest that we can pin down an intended natural number structure (and intended truth values for number theoretic claims) by talking about what *would* be true of the relevant initial segment of the emperors if the emperors satisfied PA^i .

More specifically, a multiverse theorist employing the physical explanation paraphrasing strategy considered in this section should accept the following principles.

- Metaphysical Possibility of Emperors satisfying PA^i. There is some metaphysically possible world w at which the emperors under successor satisfy PA^i.

- Metaphysical Necessity of Plenitude: At every metaphysically possible world w there is a multiverse of universes all satisfying ZFU[25] and all universes in this multiverse agree on the physical objects at w.

  Moreover, for any possible way of choosing from the physical objects at w there is a universe in the multiverse at w which contains a set s consisting of exactly those physical objects.[26] Also we can unproblematically quantify over all sets in the universes in the multiverse at w (at the same time).

So we can say the following things, which I take to conflict Hamkins' repudation of any single intended model for the natural numbers.

[25]By this I mean ZF set theory plus an axiom U for ur-lements saying that e.g. there is a set of all physical objects.

[26]That is, for all pluralities xx of physical objects in w, the multiverse in w contains some universe with a set whose elements are exactly the xx. And – although this will make no difference to the main argument in this section – the same goes for all possible ways of choosing n-tupples of physical objects (i.e., all possible ways an n-place relation could apply to physical objects in w) being witnessed by some set of sets coding n-tupples in some universe in the multiverse at w.

- (Regardless of what structures exist within pure set theoretic hierarchies) the objectively intended natural number structure is that which would be instantiated by the good emperors (defined as above) if the emperors (under successor) satisfied PA^{-}.
- An arbitrary number theoretic claim ϕ will be true under the intended interpretation of natural number talk iff it is metaphysically necessary that if the emperors (under successor) satisfy PA^{-} then the good emperors (under successor) satisfy ϕ.

What about other width multiverse theorists, who might accept a unique intended natural number structure, but reject a theory-choice independent right answer to CH (and other questions whose truth value can be changed by forcing)? A similar strategy can be used to create a sentence which (given the four assumptions above) is intuitively true iff CH is false [27].

5 Appeal to Provability in FOL

The third and final style of response to the lazy explanatory indispensability challenge which I want to consider, tries to replace claims about all possible ways of choosing (expressed via set theory) with claims about provability §2.1.

This approach can be motivated by noting a certain fact about the non-three colorability case from §2.1. (From a traditional point of view), if the

[27] In brief, we can write down a sentence that's should be true iff CH (as intuitively understood) is false i.e., iff there is some possible way of choosing some objects xx from among (objects with the intended structure of) the natural numbers, which have a cardinality strictly between $\aleph_0$ (that of the natural numbers) and $2^{\aleph_0}$ (that of the real numbers). We do this by creating a sentence which (in effect) says that the following scenario is metaphysically possible. The stars have the cardinality of the (informally intended) natural number structure, the pebbles have the cardinality of the (informally intended) powerset of the natural number structure, and the red pebbles have a cardinality strictly in between that of the stars and the pebbles.

As above, our key tool for creating writing sentence is exploit the assumption that each metaphysically possible world contains a multiverse of hierarchies-of-sets-with-ur-elements that collectively all possible ways of choosing some items (or ordered pairs, or triples of items etc) from among the physical objects in that world. This gives us the power of second order function and relation quantification (via talking about sets), which makes expressing the ideas above straightforward.

non-three colorability explanation is true, there's a version of this explanation that eliminates all appeal to sets and all possible ways of choosing. Specifically, there's a first order logical proof (which using no mathematical, second order logical or other contested notions) that the map isn't three colored from finitely many facts about which map regions are adjacent to each other[28].

My main objection to this strategy for answering the lazy indispensability challenge is that it doesn't generalize (in any obvious way) to handle other seemingly cogent physical explanatory hypotheses that invoke a notion of all possible ways of choosing/set theory, but have a more logically complex structure (like the $\forall\exists$ explanation involving possible assignments of attacking and defending troops above). There's no appearance that this more complex explanation is true if and only if some concrete first-order logical statement about adjacency and map regions entails the explanandum.

However, additional worries arise about whether Hamkins (or other multiverse theorists) can use the above strategy to replace *even* the basic three-coloring explanation above. I will discuss these extra worries in some detail, because I think they reveal interesting philosophical choice points for the multiverse theorists, although I think they are not particularly dialectically important given the stronger objection above. Specifically, we face a dilemma when trying to cash out the proposal above (even in the promising three case of the three

[28]Consider an infinite language with relations A 'is adjacent to', R 'red' G'green' B 'blue', and separate names a, b, c . . . for each tile on the map. Let T be the infinite theory which contains atomic sentences specifying A(a,b) or ¬A(a,b) for each pair of tiles (depending on whether the named tiles actually are adjacent to each other or not), plus the FOL assertion that every tile is red green or blue and no two adjacent tiles are both red, both green, or both blue. If T has a model, then the map is three colorable (you can just color each tile to mirror the way the model assigns colors R,G and B to the names a, b, c, as every subgraph of a three colorable graph is three colorable). And by completeness, if T is syntactically consistent, it has a model. So if the map is not three colorable then T is not syntactically consistent. Since any proof in FOL has only finitely many premises, there are finitely many atomic adjacency facts (using finitely many names) which jointly imply that the map is not three colored. And one could use Ramsification to eliminate these names. Thus, there is a true existential sentence (in our current, finite, language) expressing the adjacency facts about this finite region which entails the map can't be 3 colored.

coloring explanation) as follows.

5.1 Appeal to a Specific Proof

On one hand, the multiverse theorist *could* replace the traditional set-theoretic non-three colorability explanation with a specific first order logical deduction of the fact the map isn't three colored, from specific facts about which map regions are adjacent to each other. That is, they might produce an argument with the following form (where the ellipses are filled with specific deductions and facts about the map in question).

> 'That map will never be three colored because it contains countries related by adjacency like ...., hence ... so the map isn't three colored'

But this approach faces two problems. First, the explanations produced are significantly less unifying and explanatory than the original explanation by appeal to three colorability.[29]. Because it appeals to specific facts about the map in question, it doesn't bring out what this map has in common with other maps that aren't three colored because they aren't three colorable.

Second, this strategy can't be applied in all cases where we might want to propose the original non-three colorability explanation. It seems that we can entertain the conjecture (and perhaps even know) that some map is not three colored because it is not three colorable while *not knowing* any specific facts about adjacency relations on the map which entail it's not three colored[30] But

[29] In [14] Putnam famously contrasted unifying high-level explanations like 'this can't fit through that, because this is a square peg with side length ... and that is a round hole with diameter...' with the corresponding microphysical explanation one might give for the same fact. And (in the traditional explanatory indispensability literature) nominalistic paraphrases are commonly criticized for creating this kind of loss of unifying-explanatory power. See, for example, the criticisms Hartry Field's proposed infinitary nominalistic paraphrase of Newtonian mechanics in [5] in works like [4]. The strategy of replacing the non-three colorability explanations with a first order logical deduction of the fact that the map isn't the colored from specific features of the map in question (rather than using both, as someone with more traditional views of set theory can) seems to involve a similar loss in unifying and therefore explanatory power.

[30] Perhaps we would need to know such facts to *prove* that the map isn't three-colorable. But, as

the strategy considered in this subsection won't let us formulate/conceptually explicate such conjectures.

5.2 Appeal to Proof Existence/Provability

On the other hand, one can avoid both of the above problems by giving an explanation that quantifies over proofs and asserts *the existence* (or possibility) of a proof that the map is not three colored (from some true first order logical facts about adjacency relations between map tiles) – rather than giving any particular such proof. That is, we might propose an explanation along the following lines:

> The map isn't ever three colored because it is (first order logically) provable from *some* true sentences about how finitely many countries are related by adjacency that it won't be three colored.

However, I will argue that this approach is also difficult to combine with Hamkins' view — and perhaps also multiverse theory generally.

What blocks Hamkins from using this approach is (I'll suggest) the existence of prima facie close ties between intuitive concepts of provability and the intended natural number structure (which, as noted above, Hamkins rejects).

For example, Hamkins motivates skepticism about whether we can refer to a unique intended natural number structure (as opposed to alternatives traditionally considered nonstandard models) via skepticism about whether thoughts like '0, 1, 2 and so on' can secure a definite structure/stopping point for the natural numbers[31]. But this worry would seem to equally apply to our

recent work like [2] notes, we can sometimes explain physical facts by appealing to a mathematical claim whose truth we rationally suspect but haven't proved.

For example, Baker notes that scientists correctly hypothesized that bees had hexagonal honeycombs because this allowed for an optimal of side-length-to-area (under certain constraints) before they had any proof of the relevant mathematical fact (i.e., at a time when this was only a plausible/motivated conjecture).

[31] See, for example, [8]

grasp of how many stages of inference a proof can contain[32][33].

[34]

What about other width multiverse theorists? Other multiverse theorists can avoid all the specific worries for Hamkins just mentioned, just by accepting a unique intended natural number structure. I think even this move is not entirely without costs[35]. However, I won't go into detail about these problems

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[32]We think proofs can have any finite number of steps, but you can't have infinite descending chains of proof steps (corresponding to non-standard models of number theory)

[33]Relatedly, we traditionally expect that a claim is provable in some formal system iff a Gödel number coding such a proof exists. But (in addition to rejecting a unique intended natural number structure), Hamkins seems to countenance universes that disagree on the truth value of such arithmetical provability claims. So there's a prima facie worry about whether universes which get provability facts 'wrong' (e.g., making $\neg Con(A)$ claims true, in cases where contradiction is not derivable from A) qualify as wrong for reasons unrelated to mathematicians' choice of which axioms to work with (contra Hamkins).

Perhaps Hamkins could reply by saying that expected applications of set theory to provability are irrelevant to pure mathematics, like traditionally expected applications of geometry are irrelevant to physics?

Note, the issue here isn't that some set-theoretic universes could contain fake *three coloring functions* (presumably they cannot). Rather, it's that some set-theoretic universes will contain numbers corresponding to (what a traditional realist would consider to be) fake *proofs* of non-three coloring from some collection of truths about adjacency relations on the map. So, we get the following situation. In fact, whenever a map isn't three-colorable, we can (from a traditional realist point of view) derive the fact that it won't be three-colored from finitely many truths about adjacency relations on the map in FOL. But it's not clear that there's any claim that one can make about Hamkins' multiverse which expresses this provability claim

[34]In principle, Hamkins could accept that there's a standard favored notion of provability, and correspondingly some favored set universes whose natural numbers 'get provability facts right' – in the sense that something is provable iff there's a Gödel number coding a proof of it in the copy of the natural numbers in these universes. They could then formalize provability claims in scientific explanations as claims about the natural numbers in these favored universes.

Denying that there's a unique favored natural number structure does not absolutely forbid Hamkins from taking this line. For there's no direct contradiction in saying that universes can be standard or nonstandard as regards their notion of *provability* (i.e., the truthvalue they assign to specific sentences about Gödel numbers coding proofs) while denying that there's a unique favored notion of being an ω sequence/intended model of the numbers. By the completeness theorem, there will be infinitely non-isomorphic structures that satisfy the PA axioms and get these provability facts right. So one could say all of these different interpretations of 'ω sequence' (which agree on provability facts) are equally intended.

However, I strongly doubt that Hamkins (or anyone sympathetic to the strong form of multiverse theory suggested by the closure principle above) would be inclined to take this line. For one thing, he has rejected appeal to such a favored notion in conversation. Additionally, the naive/traditional notion of provability tends to include not just the idea of provability facts (facts about what's provable in a given formal system) but also determinate facts about whether various long sequences of, say, inscriptions would count as genuine proof. But the latter notion (of a structure containing only genuinely finitely many successors of zero, not extra points at infinity), is exactly the notion we attempt to appeal to when attempting to refer to the intended model of the natural numbers.

[35]For example, Hamkins (in conversation) has used the general fact that set universes can disagree on their ordinals to explain why we can't use Fregean abstraction principles (as per [9]) to introduce a single set hierarchy which (in effect) contains all sets from all universes in the multiverse, along the

because of the bigger issue mentioned at the beginning of this section. The strategy of replacing appeals to all possible ways of choosing with appeals to provability can't address the lazy explanatory indispensability challenge, because of the bigger issue (about explanations with a more complex logical structure) mentioned at the beginning of this section.

6 Conclusion

In this paper, I've presented an explanatory indispensability worry for Hamkins' multiverse theory, (and multiverse approaches to set theory in general). I then suggested a few different strategies for answering this worry, and noted some problems for each.

In doing this, I don't claim to have refuted any version of the multiverse theory. Instead, I've tried to show how accepting multiverse set theory raises an immediate question about what to say about the apparently cogency of physical explanatory hypotheses which appeal to facts about 'all possible ways of choosing' (stated via set theory). I think that in clarifying their favored answer this question, multiverse theorists like Hamkins would helpfully clarify their views about pure mathematics as well [36].

following lines. Recursively specify identity conditions for objects in this larger universe by saying that two sets x and y occurring at a given level α in different universes in the multiverse belong to the same new-set iff their elements do (c.f. Martin [11]). So a multiverse theorist who takes all universes to agree on their natural number structure faces a dilemma. If they reject Hamkins' claim that universes disagree about the ordinals, they will need to find some other way of blocking the universe-combining challenge above (perhaps by rejecting the possibility of quantifying over all sets in all universes?). On the other hand, if they say that all universes agree on the intended model of the natural numbers but disagree on ordinals at higher stages, this can seem unprincipled.

[36] For example, I haven't discussed the possibility of supplementing the official ontology and ideology of Hamkins' Platonist multiverse with an appeal to primitive modal notions (of logical possibility), when cashing out physical explanations like the story about three colorabiltity above. I don't discuss this option because it's sufficiently different from the philosophical position Hamkins takes in [6]. However I think that some such modality-centric approach to set theory is ultimately the way to go, and I discuss how Hamkins could adopt a version of it in REDACTED.

A Hamkins' Multiverse

A.1 The Multiverse

In [7] Hamkins describes his multiverse proposal as a form of Platonism, which accepts the existence of many different set theoretic hierarchies (with equal mathematical status) rather than one unique intended hierarchy of sets. On Hamkins' view, certain set theoretic statements like the Continuum Hypothesis (i.e., the claim that there is no set intermediate in size between the real numbers and the natural numbers) are not true or false simpliciter, but merely true in some parts of the multiverse and false in others. In some universes in the multiverse CH is true and in others it is false, and there is no unique intended universe. Thus (as Hamkins vividly explains in the passage below) CH cannot be settled by finding intuitively compelling new axioms from which it can be proved or refuted. For mathematicians' experience reveals there are parts of the multiverse in which CH holds and parts in which $\neg CH$ holds.

> "[If some obviously true seeming mathematical axiom] ϕ were proved to imply CH, then we would not accept it as obviously true, since this would negate our experiences in the worlds having ¬ CH. The situation would be like having a purported 'obviously true' principle that implied that midtown Manhattan doesn't exist. But I know it exists. I live there. Please come visit! Similarly, both the CH and ¬CH worlds in which we have lived and worked seem perfectly legitimate and fully set-theoretic to us, and because of this, any [proof from ϕ that CH or that ¬CH] casts doubt on the naturality of ϕ. [7]

Three further features of Hamkins' multiverse are worth noting here.

First, (I take it) Hamkins isn't proposing any kind of supervaluationist theory on which ordinary set theoretic claims are determinately iff true in every set theoretic universe and determinately false iff in every such universe (so the facts above show that CH is *indeterminate*). The idea is that set theorists study different set theoretic universes in different contexts (as well as studying the relationships between them), like historians study different cities on earth. We don't say that it's indeterminate whether 'the city' has a population larger than 4 million, but rather by saying that there are many different cities, some of which have and others of which lack this property, and we must evaluate a historian's claim by determining which city they are talking about in a given context.

Second, Hamkins' proposal is inspired by a controversial interpretation of a mathematical technique called forcing. Hamkins suggests that for each set theoretic hierarchy V satisfying the ZFC axioms, we should accept that there is another (strictly wider) set theoretic hierarchy $V[G]$, the forcing extension of V. This expanded universe $V[G]$ adds a set G to V, where G is subset of a set (a partial order $\mathbb{P}$) that's already in V — along with other sets, as needed for $V[G]$ to satisfy the ZFC axioms.

A version of this claim is uncontroversially true; if we work in some background notion of set theory we can prove that that every *countable model* of the ZFC axioms for set theory has a forcing extension (as mainstream/conventional approaches to forcing do)[37]. In contrast, Hamkins endorses the general claim that every set-theoretic hierarchy satisfying the ZFC axioms has a forcing extension. He thus contradicts traditional/mainstream views that the intended hierarchy of sets already contains 'all possible' subsets all sets it contains, – so there can be no extended universe V[G], which adds a missing subset to a set (the partial order $\mathbb{P}$) this V already contains.

[37]I omit discussion of Boolean valued models and inner models, as other possible routes to understanding forcing arguments without admitting that our background set-theoretic universe could be widened in the way described above.

Finally, Hamkins asserts more powerful principles than the above claim about taking forcing extensions. Specifically, he makes the provocative claim that, "Every universe V is ill-founded from the perspective of another, better universe."[6][38] (while no process of repeatedly taking forcing extensions can generate such a universe). Note that, in such cases, the natural numbers in V will be non-standard from the perspective of V', meaning that different set-theoretic hierarchies can have different views about what number theoretic claims are true.

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