The Residual Access Problem

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Introduction: The Access Problem

Truthvalue Realist Intuition: Mathematical facts can outstrip provability (e.g. there are definite right answers to all questions in the language of arithmetic).

Access Problem for Mathematical Truth-Value Realists (roughly): If there are objective mathematical facts (which outstrip provability), how could our possession of substantial mathematical knowledge be anything but a miracle or a mystery?

The Access Problem

I will say that: A truth value realist philosophy of mathematics **faces an access problem** to the extent that combining it with uncontroversial scientific and philosophical facts

- (apparently) commits one to the existence of *some* ‘extra’ match between our mathematical beliefs and belief-independent facts which intuitively cries out for explanation, but goes unexplained.

Apparent Commitment to A Spooky Coincidence

If this appearance is correct, it provides some reason to reject truth value realism.

- Ceterus paribus we want to avoid theories committed to positing such ‘spooky coincidences’, i.e., regularities which cry out for explanation but don’t get explained

The Structuralist Consensus

I'm a truth value realist. What to do?

A popular Structuralist Consensus suggests a place to start. It implies we can reduce

- the mathematical truth value realists' access problem to
- an access problem re: knowledge of logical coherence.

Agenda

In this talk I'll

- Review how the structural consensus lets us reduce the access problem to a simpler access problem problem.
- Propose a solution to this residual access problem[1].

[1] Here I extend my proposal in [2].

Outline

1. Introduction: The Access Problem
2. Reduction to An Access Problem for Logical Possibility
3. Knowledge of Logical Possibility of First Order Claims
4. Possibility of Non-First Order Claims
5. Objections

Reduction to An Access Problem for Logical Possibility

The Structuralist Consensus pt1: Mathematicians' Freedom

Structuralist Consensus pt1: Mathematicians can introduce (almost) any "logically coherent" stipulations defining a pure mathematical structure they wish,

- Note that I'll take this notion of logical coherence/possibility $\Diamond$ as a conceptual primitive not reducible to facts about either syntactic derivability or set theoretic models.

Mathematicians' Freedom

This idea is motivated by mathematical practice

Mathematicians' Freedom:

> *Reflecting on my experiences as a research mathematician … [I was struck by] the freedom I felt I had to introduce a new mathematical theory whose variables ranged over any mathematical entities I wished, provided it served a legitimate mathematical purpose. [3]*

Ways of Allowing Mathematicians' Freedom

Examples of how different philosophers allow this

- Plenitudinous Platonists and Neo-Fregeans: mathematical universe is so large that (almost) all coherent math posits are satisfied somewhere.
- Modal Structuralists: reduce structure talk to modal talk $\Diamond D \& \Box(D \rightarrow \phi)$
- Quantifier Variantists: coherent math posits change the meaning of '$\exists$' (among other things) to ensure their own truth.

Structuralist Consensus: Logically Powerful Posits

Structuralist Consensus pt2: These pure posits can categorically describe structures like the natural numbers and thereby pin down proof transcendent facts about them - as the truth value realist desires.

- e.g.. the categoricity of PA_2[2] ensures that for every sentence ϕ in the language of arithmetic
  - either ϕ or $\neg\phi$ will be a logically necessary consequence of PA_2.
  - i.e., $\Box(PA_2 \rightarrow \phi)$ or $\Box(PA_2 \rightarrow \neg\phi)$

[2] The second order Peano Axioms

Access Problem Reduction

On the Structuralist Consensus

- there is no mystery about how we decided to work with the 'real' mathematical objects and not fake ones.
- Accordingly we can reliably gain mathematical knowledge by
  - Introducing logically coherent axioms characterizing new pure mathematical structures.
  - Deriving logical consequences from these axioms.

All done? No

But this doesn't solve the access problem. For the ability to recognize coherent posits can itself seem mysterious.

- How can mathematicians recognize that, e.g., they can coherently posit PA_2 but not 'PA_2 and there are a finite number of primes'?

Residual Access Problem

So it looks like we could solve the access problem if we could solve the following

Definition (Residual Access Problem)

How could mathematicians have acquired the ability to

- recognize (enough) logically coherent posits
- derive (enough) logical consequences from them?

(without benefiting from some coincidence less realist philosophers of math needn't posit)

I'll answer by providing a toy model.

A 'How Possibly' Question

For I take access problems to involve a 'how possibly/how plausibly' question:

- How could mathematicians possibly have acquired the ability to recognize (enough) logically coherent posits without benefiting from some spooky extra coincidence (given certain general facts about human nature etc.) ?
- c.f. How could demand for potatoes possibly have increased with the price of potatoes during the Irish Potato famine (given certain general facts about human nature etc.)?

Gold Standard: Answer by Toy Models

'How possibly' questions are most directly answered by providing a toy model which:

- fits all facts that drive the impossibility intuition
- but may illuminatingly simplify/ignore other details

e.g. We can answer the potato famine 'how possibly' question by a story w/ round number prices and simplified food preferences (with potatoes as cheapest food and higher quality foods people cease being able to afford) .

What We Want

Want: toy model that does the same for the residual access problem.

To this end I'll tell a simplified story about how creatures

- initially (just) able to reliably acquire true beliefs in a first-order language by FOL deduction, perception, IBE etc.
- could have come to reason well about logical possibility (e.g., $\Diamond\phi$ claims[3])

(without benefiting from extra mysterious coincidences).

[3] Note mere use of FOL will never tell you anything is logically possible

Agenda

- First I'll consider reasoning that yields $\Diamond\phi$ claims where ϕ is first-order.
- Then I'll extend this picture to account for such knowledge when ϕ is in a richer logical language able to categorically describe structures like the natural numbers.

Knowledge of Logical Possibility of First Order Claims

Setting The Stage

- Creatures thinking in and speaking an FOL language can represent many things that won't occur because they are logically impossible.
- So (absent logical possibility knowledge) they risk wasting time trying to
  - realize logically impossible plans
  - counter logically impossible dangers

It would be advantageous for creatures like these to get good methods of reasoning about logical possibility **if they can**.

My Proposal

I'll suggest that

- Scientific induction-like generalization[4] from patterns in what actually happens could lead us to good methods of reasoning about logical possibility.
- Thus, our knowledge of logical possibility is ultimately no more mysterious than our our knowledge of physical or chemical possibility.
- Deploying good methods of reasoning about logical possibility can let us recognize coherent mathematical posits.

[4] whether at the level of evolution, cultural selection or individual experience

Mechanisms

We can think of this in terms of several mechanisms of correction and improvement, which could have lead such creatures

- from good FOL reasoning, perception, IBE etc.
- to being disposed to make **largely correct rather than incorrect** judgments about logical possibility and impossibility[5].

[5] insofar as they are inclined to make any judgments at all

Initial Datapoints ϕ to $\Diamond\phi$

Knowledge of concreta gives us initial datapoints regarding logical possibility.

- We can recognize that a state of affairs is logically possible by seeing that it is actual.
- Note that it's enough to show that *any* instance of ϕ is actual.
- e.g. If you aren't sure whether $\Diamond\phi$ is true where ϕ is written in terms of some relations F, N, M.
  - noting that that friendship, nephew-hood and having been in military service apply in this way to the royal family of Sweden will convince you of $\Diamond\phi$

Generalization to $\Diamond\phi$ Deriving Rules

Learning such $\Diamond\phi$ can correct bad reasoning methods which let us derive $\neg\Diamond\phi$

Abduction from these data-points can yield general principles and reasoning methods which let one recognize additional $\Diamond\Phi$ claims. These can be:

- inference rules[6] e.g.,ones of the form 'if $\Diamond\Phi$ then $\Diamond\Psi$' 'every logically possible scenario which... could be modified to create another one in which...')
- ways of using mental pictures
- subpersonal mechanisms

[6] My draft book 'A logical foundation for potentialist set theory' has an example of such an inference system.

Selection and $\Diamond\phi$ Deriving Rules I

Something similar could happen at the level of natural selection.

Ruling out actual (or practically achievable) states of affairs as logically impossible can cause practical trouble, e.g.,

- rejecting a description of the enemy's actual plan of attack as logically impossible to satisfy, and hence failing to counter it.
- failing to implement some physically possible and mutually satisfactory division of spoils.

And similarly being able to recognize (and hence prioritize) hypotheses which are at least logically possible can be useful.

Selection and $\Diamond\phi$ Deriving Rules II

So there could be selective pressure to

- drop bad methods of reasoning which let one derive that various physically possible things are logically impossible.
- add methods of reasoning which let one recognize more physically possible states of affairs as logically possible (in advance).

$\neg\Diamond$ Knowledge I

What about logical impossibility knowledge?
Classing claims as logically (not just physically) impossible can provide the best explanation for[7] patterns in first order concrete states of affairs.

e.g. Suppose someone thought it was logically possible for 5 people to choose different combinations of toppings from a sundae bar with two toppings.

- They'd have to somehow explain the striking regularity that, regardless of the type of items and properties, we never wind up observing more than 4 such objects.

[7] and most efficient way to predict

$\neg\Diamond$ Knowledge II

One *could* postulate new physical or metaphysical laws to explain why apparently free choices of sundae toppings never generated the forbidden 5th possible outcome.

But note these laws would *also* have to explain why the analogous regularitiesheld, in exactly the same way

- at every physical scale we can observe, from relationships between the tiniest particles to relationships between planets and stars
- for much less concrete subject matter like poems or countries.

$\neg\Diamond$ Knowledge III

Bigger picture: We attempt to anticipate constraints on what's practically possible via some combination of:

- general facts about what is logically possible,
- specific metaphysical/analytic facts about the properties and relations involved
- contingent scientific laws.

Sometimes considerations of theoretical elegance (and practical efficiency) will favor explaining a particular constraint in terms of logical impossibility.

Reflective Equilibrium

Once we have some initial good methods we can

- correct false generalizations by noting when they conflict with better-entrenched and concretely motivated principles.
- IBE lets us further generalize

  There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems [that] they would have to be accepted at least in the same sense as any well-established physical theory [4].

Power of Generalization

The kind of elegant scientific generalization/IBE I want to invoke goes beyond simple inferences like ‘the sun rose every day for the past billion years, so it will rise tomorrow.’

- Observations of points of light in the night sky can lead to an elaborate model of how the planets are arranged.

Knowledge of Logical Possibility

So I've suggested that a combination of

- Access to first order truths about concrete objects
- Knowing that (all substitution instances of) what's actual are logically possible (making the inference from Φ to $\Diamond\Phi$).
- Inference to the best explanation

could have given creatures (idealized versions of us) accurate methods of reasoning about the logical possibility of first order states of affairs.

Possibility of Non-First Order Claims

Possibility of Non-First Order Claims

Recall, truthvalue realism needed categorical posits (like PA_2) to pin down proof transcendent facts.

- No FOL claims can do this.
- Hence we must explain $\Diamond\Phi$ knowledge for Φ a sentence in a stronger logic.

Failed Idea

First thought **use second order logic**.

- If we could (somehow) presume knowledge of some basic statements Φ involving second order quantification
- then maybe we could use the story about generalization above to explain knowledge of $\Diamond\Phi$ claims (e.g. $\Diamond PA_2$)

Problem: We can't just assume knowledge of second order logic without risking begging the question.

Access to Second Order Objects

Many would say:

- $(\exists F)(\forall x)$(F(x) iff x is a brown egg) requires the existence of a second order object.
- Knowledge of (abstract causally inert) second order objects can seem mysterious in exactly the same way knowledge of sets would be.
- It's not like we can just "see" second order objects. We don't see sets of eggs floating over an egg carton.

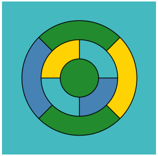
Generalizing the notion of $\Diamond$

Solution: Develop an (similarly powerful) notion of "**conditional** logical possibility" $\Diamond_{R_1,\ldots R_n}$ which

- has all the expressive power of second order logic
- is so similar to logical possibility simpliciter ($\Diamond$) that
  - we can **explain** initial knowledge of conditional logical possibility by appeal to the mechanisms of elegant generalization that gave knowledge of $\Diamond$ facts above.

A Motivating Example: Three Colorability

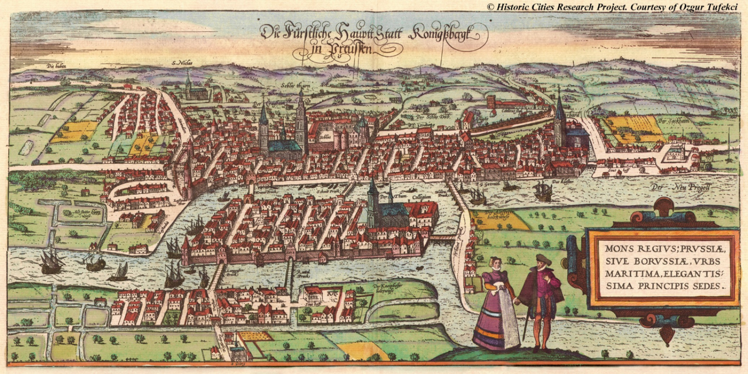
Suppose we have a map like this:



It's logically impossible, given the facts about how 'is adjacent to' and 'is a country' apply to the countries on this map, that each country is either yellow, green or blue and no two adjacent countries are the same color.

A Motivating Example: Köeningsburg Bridges

We can also think about the famous property of the Königsberg bridges (that there's no way of traveling over each bridge exactly once) in these terms.



How $\Diamond_{R_1...R_n}$ Generalizes the notion of $\Diamond$

When evaluating logical possibility $\Diamond$ we:

- ignore all limits on the size of the universe
- consider only the most general combinatorial constraints on how any relations could apply to any objects (c.f. Frege).
- ignore subject matter specific constraints so, e.g., $\Diamond\exists x(Raven(x) \wedge Vegetable(x))$ comes out true.

Likewise for conditional logical possibility $\Diamond_{R_1...R_n}$. But we also hold fixed the application of certain specific relations $R_1...R_n$

Conditional Logical Possibility

I'll use the notation $\Diamond_{R1...Rn}$ to express claims about what is logically possible **given the facts about how certain relations apply.** Consider:

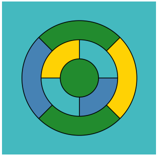
> **C&B**: *'It is logically impossible, given what cats and baskets there are, that each cat is sleeping in a basket and no two cats are sleeping in the same basket.'*

There's an intuitive sense of 'logically impossible' on which this claim will be true iff *there are more cats than baskets* in the actual world.

I'd write this as $\neg\Diamond_{cat,basket}$ [Each cat is sleeping in a basket and no two cats are sleeping on the same basket.]

More Complex Conditional LP Claim

We can express the three colorability claim as follows



$\neg\Diamond_{adjacent,country}$ Each country is either yellow, green or blue and no two adjacent countries are the same color.

Note: I'm proposing conditional logical possibility as a conceptual and metaphysical primitive, but one can relate it to familiar set theoretic notions.

Example of Replacing Second Order Quantification

We can use this notion to express claims like second order induction:

$$(\forall X)\,[(X(0) \wedge (\forall n)\,(X(n) \rightarrow X(n+1))) \rightarrow (\forall n)(X(n))]$$

- **Induct**: '$\Box_{\mathbb{N},S}$ If 0 is happy and the successor of every happy number is happy then every number is happy.

It is logically necessary, given how $\mathbb{N}$ and S apply, that if 0 is happy and the successor of every happy number is happy then every number is happy.'

Nesting Conditional Logical Possibility I

We can also make claims about the logical possibility or impossibility of claims like **C&B** , saying things like

- It would be logically possible for 'cat' and 'basket' to apply in such a way that it would be logically impossible, given what cats and baskets there are, for each cat to sleep on a different basket.
- $\Diamond(\neg\Diamond_{cat,basket}$ each cat is sleeping in a basket and no two cats are sleeping on the same basket.')

Nesting Conditional Logical Possibility II

$\Diamond(\neg\Diamond_{cat,basket}$ *each cat is sleeping in a basket and no two cats are sleeping on the same basket.')*

This claim, $\Diamond$(C&B), is true because:

- It's logically possible (holding fixed nothing) that there are 4 cats and 3 baskets.
- Relative to the scenario where there are 4 cats and 3 baskets, it's not logically possible, given what cats and baskets there are, that each cat slept on a basket and no two cats slept in the same basket.

Sufficiency of Descriptions

- We can write a sentence $PA_{\Diamond}$, (purely in terms of logical possibility) which categorically describes the natural numbers[8].
- In [1] I argue we can similarly rewrite other second order conceptions of pure mathematical structures.
- So, to answer access worries it suffices to account for knowledge of claims like $\Diamond \mathrm{PA}_{\Diamond}$.

I propose that...

[8] Just use the fact above to replace the second order induction axiom with a conditional logical possibility claim.

Accounting for More Knowledge

- The same method of generalization explained above gives us (some) knowledge of $\Diamond_{R_1,\ldots R_n}\phi$ and $\Box_{R_1,\ldots R_n}\phi$ claims for ϕ first-order.
    - e.g., the best explanation for the fact that no one ever crosses all the Köningsburg bridges exactly once is that it would be logically impossible to do so, given the connectivity facts.
- Applying this method again gives methods of reasoning which produce knowledge of claims like $\Diamond \mathrm{PA}_{\Diamond}$.

Summary

I've given a toy model explanation for how how creatures like us could have acquired (sufficiently) good methods of reasoning about logical possibility by

- inference from ϕ to $\Diamond_{R_1 \ldots R_n}\phi$
- considering substitution instances.
- (something like) abduction/IBE.

On the Structuralist Consensus math knowledge follows by

- Adopting any logically coherent axioms characterizing pure math structures they like.
- Deriving logical consequences from these axioms.

Objections

Worry 1: Reliability of Scientific Induction

Some doubt that scientific induction is ever valid in mathematics (i.e., can't be confident of a mathematical hypothesis just by checking enough cases)
But,

- Mathematicians frequently use hunches, past experience and the results of computational searches to guide their research.
  - Belief in Fermat's last theorem was largely the result of a consistent failure to find a counterexample.
- If we take this practice seriously we can't totally reject elegant generalization from cases in mathematics.

Worry 2: Gap Between Finite and Infinite

Worry 2: Maybe this kind of generalization doesn't extend to the infinite.

Response:

- The physical world seems to be *(at least) helpfully describable in terms of* some infinite collections, e.g., spatiotemporal points etc.
- Plausibly this can create pressure to acknowledge the logical possibility of certain kinds of (small) infinite systems, and to avoid unreliable reasoning about what these systems must be like.

Worry 3: Fading out

Maybe generalization from small collections can't support *enough* knowledge of logical possibility:

- Our dealings with objects in the world tend to involve finite (or relatively small infinite) collections gingerbread cookies, portions of space along the path of an arrow, etc.
- Yet, providing a nominalist paraphrase for statements of set theory requires evaluating claims about the logical possibility of scenarios involving vast numbers of objects.

A critic might advance the following analogy:

> *Saying that elegant generalization from knowledge of finite and countable collections yields principles which accurately describe the larger collections considered in pure mathematics is like saying that IBE plus observations of birds in New Mexico allows us to learn about birds in Canada as well.*

I accept this analogy, and claim that it actually fits the current state of human knowledge with regard to facts about the higher infinite rather well.

- We can know *some* things about birds in Canada just by IBE from the facts about the birds in New Mexico e.g. we would expect them to have DNA, hollow bones etc.
- Our expectations about birds in distant locales are just relatively **sparser** and **less confident** than our beliefs about birds in locations we've observed.

But this is just what happens with regard to our knowledge of what's logically necessary with regard to large collections:

As one goes from claims about finite collections to countable collections (like the numbers), to uncountable collections (like the sets) mathematicians' beliefs **do** appear to get

- sparser, e.g., the continuum hypothesis (CH) is a fairly simple question involving sets of (relatively) small infinite size, yet it is known that both the truth and the falsity of CH are compatible with ZFC.
- less confident: mathematicians are more confident in their claims about numbers, sets of numbers and sets of sets of numbers than in distinctive claims of higher set theory claims about much larger structures.

Thus, I think the above worries about size actually point to a benefit rather than a flaw of the account at hand: it predicts (and thereby helps explain) the way that our knowledge of the mathematical objects does appear to peter out.

Conclusion

Conclusion

In this presentation I've suggested we can dispel access worries about mathematical truth-value realism by

- adopting some view (in the Structuralist Consensus) on which mathematicians are free to adopt logically coherent pure mathematical posits.
- providing a toy model of human accuracy about logical possibility in a suitably powerful language.

Natural Selection and Abduction

Worry: Can evolutionary selection to efficiently predict cases do realiable (rather than grue-some) generalization from cases, something like IBE?

- We mostly presume it can.
- e.g., If babies turned out to be relatively hardwired to correctly judge which foods would make them sick or which dogs were about to bite this wouldn't create access worries.

Mimicing Nested Relative Logical Possibility

Using set theory, we can approximately mimic truth conditions for claims about nested logical possibility as follows...

> *Let ϕ be a formula with no free variables. $\Diamond_{R_1,\ldots,R_m}\phi$ is true relative to a model $\mathscr{M}$ just if there is another model $\mathscr{M}'$ which assigns the same sets of tuples[9] to the extensions of $R_1,\ldots,R_m$ as $\mathscr{M}$ and makes ϕ true.*

ϕ is true full stop if it is true relative to the model/interpretation $\mathscr{M}$ which assigns all nonmodal vocabulary standardly.

[9] Remember, the extension of an n-ary predicate R is the set of tuples $\langle a_0,\ldots,a_{n-1}\rangle$ such that $R(a_0,\ldots,a_{n-1})$ holds.

$\Diamond_{R_1,\ldots R_n}$ Preserves Shapiro's Structure

- system: some objects considered 'under' some relations $R_1 \ldots R_n$ which apply to them [6]
- structure: 'the abstract form' of a system got by "highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system."
- e.g. "The natural-number structure is exemplified by the strings on a finite alphabet in lexical order, an infinite sequence of strokes, an infinite sequence of distinct moments of time, and so on."

So we might say: $\Diamond_{R_1,\ldots R_n}$ holds fixed **what structure** the system of objects related by $R_1 \ldots R_n$ (considered under the relations $R_1 \ldots R_n$) instantiates.

Ingredients list

Note that in the story above I'm just assuming we can use

- abduction/IBE : may have its own access worries but mathematical truth value anti-realist already accepts it
- minimal starting knowledge about logical possibility which -for most people I've consulted- doesn't seem puzzling, i.e., knowledge that $\Diamond$ is
  - A kind of possibility so $\Phi \vdash \Diamond\Phi$ (and $\Box\Phi$ iff $\neg\Diamond\neg\Phi$)
  - Possibility in re: logical structure, so substitution instances of what's logically possible are logically possible, i.e., $\Diamond\Phi$ iff $\Diamond\Phi[R_1/R'_1 \ldots R_n/R'_n]$

  (and analogous knowledge for $\Diamond_{R_1 \ldots R_n}$)

Conditional Possibility And Set Theory

Potentialist set theory (Putnam, Parsons etc.): avoid Burali-Forti arbitrariness worries re: height of hierarchy by cashing out set theory as claims about how hierarchies of sets could be extended. Using $\Diamond_{R_1,\ldots R_n}$ (rather than second order quantification) lets us

- Simplify existing formulations of potentialist set theory e.g. avoid quantifying in
- Justify (potentialist versions of) ZFC axioms from obvious seeming principles
  - Avoiding quantifying in lets us avoid controversies re: normal modal FOL, essences
- Thereby solve a problem for everyone about justifying Replacement

Usefulness: Indispensibility & More

Indispensibility: Uniformly nominalistically regiment all scientific talk satisfying a certain Definable Supervenience condition.

- thereby Reduce Quinean Indispensibility challenge to a metaphysical 'factoring' challenge (in a way I think is illuminating)

Also WIP:

- Quantifier Variance: Systematically describe truth conditions for sentences in more 'ontologically profligate' languages without paradox.
- Possible Worlds (Access and Modal Erzatses) : Create a 'potentialist' version of the structure of Chalmersian 2-D metaphysically possible worlds?

AW C 1: On Certainty

Watkins raises the following objection.

- Mathematical knowledge requires certainty.
- My account can't allow for justified mathematical certainty (because it suggests that we know math claims by IBE/scientific induction)

I'll argue against both claims.

Mathematical Knowledge Requires Certainty? I

Cases of apparent mathematical knowledge without certainty:

- can know restaurant tip but we'll recalculate it if someone disagrees
- question theorems if experts disagree (c.f. Kitcher [5])
- as time passes w/ no mistakes found confidence goes up until sufficient for knowledge. (Clay prize only given 2 years after publication)
- can know Axiom of Choice, even if we shouldn't be completely certain because of the Banach-Tarski paradox.

Mathematical Knowledge Requires Certainty? II

> *[G]enuine mathematical knowledge [should] be of propositions which we believe with a credence of 1 (or 0), since we believe that mathematical propositions are logically necessary (or impossible).*

- The fact that **philosophical** claims are metaphysically necessary (if true) doesn't make us assign them either probability 1 or 0.
- So why should the fact that mathematical claims are metaphysically/logically necessary if true tempt me to assign them probability 1 or 0?

My Explanation Prohibits Certainty? I

> *"[M]athematical or logical proofs...should make us completely certain of the result [but] if mathematical statements are known on the basis of induction, then we cannot be one hundred percent certain of them"*

I don't think we know (many) mathematical claims on the basis of scientific induction, though maybe a sufficiently long lived person could reconstruct all our knowledge that way.

Note: Explaining Accuracy vs. Justification

So far, I've not said much about knowledge or justification.

- For, following Hartry Field, access worries are often phrased as a demand to explain (mere) mathematical accuracy.

But, of course, I do think we have mathematical justification and knowledge too! So I'll say something about that now.

My Explanation Prohibits Certainty? II

I think anyone who is lucky enough to find true logical principles (including logical possibility principles) 'a priori obvious'[10] has defeasable warrant for believing them.

- So we needn't have (only or any) inductive reason for believing these principles
- We're justified a priori in accepting whatever correct basic axioms and inference procedures we find a priori obvious
- maybe it's even justified to assign probability 1 to some of them

[10] i.e. feel that they can be used without further justification unlike memories of state capitals

My Explanation Prohibits Certainty? III

Can my mechanisms account for true beliefs which feel a priori obvious? I see no reason why not. Note:

- the evolutionary version of my story could apply to biological bases for inference/belief dispositions which are as hardwired, modular and individually unrevisable as you like.
- epistemic stockholm syndrome: people accept empirical evidence re: which analysis of the Monty Haul Problem is correct, then think ‘this is how I should have reasoned a priori’.

AW C 2

> *Lack of counterexamples in the actual world should only get us something resembling ... physical necessity. Logical necessity should require something different – we need to be able to see that there could not be any counterexamples, no matter what our world was like. It is hard to see how we would ever come to know this just by examining our actual world.*

I alternated between reading this two ways (I suspect neither is quite right)

v. 1 Poverty of the stimulus

A general ‘poverty of the stimulus’ skepticism about inferring from what actually happens to necessity claims would rule out knowledge of

- non-Humean physical possibility
- non-Humean objective probabilities [11]

as well as logical knowledge, so I reject it.

[11] And I think we have to be non-Humean about objective probabilities.

v. 2 Knowledge of Metaphysical Necessity?

Maybe the question is: how can we account for knowledge that logically necessary truths are metaphysically necessary (i.e hold however the world could have been)?

- In general there's an access problem for metaphysical possibility (how can we know *anything* is metaphysically necessary?) which I haven't tried to answer here.

AW C 3: Harms of Not Recognizing Logical Necessity

> *[W]hy would thinking that a statement was [merely] physically necessary when it was in fact logically necessary ever have a negative, pragmatic effect[?]*

Bad pragmatic effects of

- Not believing $\Box\Phi$: You won't expect substitution instances of ϕ to be true (e.g. versions of it that talk about lions rather than cookies) so you will have to re-derive[12]/empirically learn them, which could be costly.
- Believing that $\neg\Box\Phi$, could prevent correct generalizations about what's logically (and hence physically) necessary.

[12] by completeness all logically necessary FOL truths are theorems, but this doesn't generalize

AW C 4: Differences between $\Diamond$ vs. $\Box$ Knowledge

> *[What are] the possible differences between knowledge of logical possibility and knowledge of logical impossibility or necessity – and [do] these differences pose any problems for [Berry's view?]."*

The two are closely related (unsurprizingly given $\Box\phi = \neg\Diamond\neg\phi$)

- only difference: we get some simple $\Diamond\phi$ knowledge directly from 'actual to possible' inferences
- abduction is needed to get
  - general inference rules/laws needed to justify more complex $\Diamond\phi$ claims
  - any knowledge of logical necessity.

Why is logical necessity knowledge needed?

> *"[Does] Berry .. need to account for logical necessity/impossibility at all[? The access problem only concerns] how we would know [that certain] mathematical structures are logically possible."*

Necessity is used to categorically describe structures, e.g.,

- Induction has the form $\Box_{\mathbb{N},S}\phi$
- So knowing that $PA_{\Diamond} = PA+$Induction is coherent means knowing something of the form $\Diamond(\psi \wedge \Box_{\mathbb{N},S}\phi)$

Also necessity claims can figure in the bootstrapping/reflective equilibrium above. Out powerful reasoning methods include both ways to infer possibility and impossibility.

OB Comment 1

C1: Assuming classical logic? Maybe we shouldn't.

Presentationally I've been assuming classical logic, for simplicity.

Note: access problems involve challenge the mathematical realist to dispel impression that the realist can't **internally coherently** explain accuracy about realist math without leaving some extra coincidence **that the anti-realist needn't posit**.

- So details re: realist and their opponent can matter.

The Simple Case I've Focused On

I've considered a realist and objector who both take for granted

- truth of classical FOL
- human accuracy about many and varied FOL claims

And suggested a way the realist could account human accuracy about for much more:

- accuracy about (something like) second order logic
- accuracy about logical possibility of non-actual approx. second order states of affairs. Including things like knowledge of $PA_{\Diamond}$

Doing this seemed hard enough!

Rejecting classical logic I

OK but what about philosophers who allows for violations of classical logic in cases of

- vagueness
- truth predicate shenanigans (e.g. the liar)?

Opt 1: They can accept the story about aprox. human reliability re: $\Diamond$ I propose, but disagree about what the general laws of logic delivered by IBE and actual to possible etc.

- c.f. People who disagree about physics can agree on broadly the kinds of things which give us access to physics (just disagree slightly re: what IBE supports).
- note: I just gave $\neg\Diamond(\exists x)(Red(x) \wedge \neg Red(x))$ as an example of something many take to be logically impossible not part of a definition.

(but philosophers who take this route might not be truthvalue

Rejecting classical logic II

Opt 2. If they want classical logic to apply to math (or only think classically coherent mathematical posits are acceptable) they can introduce a $\Diamond^C$ which means

- possibility w.r.t general laws of how objects can be related by any non-vague relations in a fragment of language without a truth predicate
- *rather than* possibility w.r.t. the most general subject matter neutral laws of how any objects can be related by any relations.

Given access to which claims can be harmlessly thought of as true in some non-vague, truth predicate free variant on our language, my story can explain knowledge of $\Diamond^C$..

Constructive mathematics

I think it's a feature not a bug that this story suggests classical and contructive mathematicians (often) substantively disagree about something ($\Diamond$)!

- Isn't that what they'd say?
- note: one can allow that even if constructivists are wrong they're still learning about something mathematical by appeal to classical reinterpretations of constructive talk.

OB Comment 2

> *Useful concepts, such as those involved in idealizations (infinitely deep oceans, perfectly frictionless planes, etc.), may be entirely misguided as accurate descriptions of the actual world [so] the reliability of conditional logical possibility must be established some other way.*

Two ways of thinking about the challenge

- legitimacy of concept of (conditional) logical possibility
- explain/defend reliability of our reasoning about what's (conditional) logically possible.

But maybe this makes little difference as e.g., illegitimate concepts are ones that make false claims analytic.

Note re: usefulness

Note: I've only talked about usefulness because I'm trying to tell an IBE story and evoke a corresponding natural selection story at the same time.

- Encountering some scenario Φ with lions can
  - Cause a person to reject an inference method which let them derive $\neg\Diamond\Phi$
  - Prevent one copy of genes that hardwire reasoning methods that lets one derive $\neg\Diamond\Phi$ from replicating itself.
- the evolutionary version of IBE requires the laws 'learned' to be useful (cf. hardwired knowledge of safe to eat foods vs. astronomy).

Nothing deep/controversial is intended!

Useful but Wrong? 1

Core Worry: useful/empirically adequate and explanatory theories can involve a bunch of false stuff, e.g., existence claims about infinitely deep oceans and frictionless planes

- Such existence claims are explicitly marked by our overall scientific theory as not literally true (only to be used in limited circs.),
- Otherwise this theory would not be so close to empirically adequate or so useful
- In contrast, a claim like 'The reason why no one has ever three colored the map above is that it's not three colorable.' (prima facie) seems literally true.

Useful but Wrong ? 2

Follow up: Easy Road nominalists say mathematical existence claims are useful for stating theories and not flagged as fictions, but not supported by IBE.

- Maybe our (conditional) logical possibility claims are like that?

Maybe I can't go beyond appeals to intuition.

- It can seem positively mysterious/anomalous how the existence of math objects could help explain, since this doesn't cause or constrain concrete events.
- In contrast deriving ϕ from logically necessary laws seems to help explain why ϕ in whatever way the fact that ϕ is physically/chemically necessary laws would.

Useful but Wrong ? [Speculative]

But I'm tempted to say more: math objects often seem dispensable because a modal version is equivalent or better, e.g., contrast

- CLP: The map wasn't 3 colored because it's logically necessary, given how the countries are related by adjacency that...
- PLAT: The map wasn't 3 colored because there is no function which takes countries to numbers 1-3 such that...
- PLAT only explains given a background belief relating math objects to modality: there are functions 'witnessing' all logically possible ways of pairing countries with numbers[13].
- whereas CLP states the modal fact that does the work directly.

but obviously I'm biased about conditional logical possibility!

[13] and hence all logically possible ways for the countries in a map w/ that structure to be red green or blue

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