Hamkins' Multiverse and Applied Mathematics

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Abstract

Set theorist Joel Hamkins' influential multiverse proposal rejects the
idea that there's a intended width hierarchy of sets which contains 'all
possible subsets' of the sets that it contains. In this paper, I raise an
explanatory indispensability worry for the multiverse theorist and discuss
several possible responses to this worry.

1 Introduction

On a conventional understanding of set theory, there's an intended model for set
theory (up to isomorphism) which contains, at each layer, sets corresponding
to 'all possible ways of choosing' sets from lower levels[1]. This determines a
unique intended right answer to all set theoretic questions, like the continuum
hypothesis (CH), whose truth value only depends on the width of the hierarchy
of sets.

Set theorist Joel Hamkins' influential multiverse program accepts the exis-
tence of sets, but denies that there's a unique intended hierarchy of sets (even up
to width) or a right answer to questions like the Continuum Hypothesis. Instead
he proposes that there's a multiverse of different hierarchies of sets (set theoretic
universes) with equal mathematical status – and that none of these universes

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[1] limit ordinals note here

satisfy the conception above, because for each universe there's a strictly 'fatter' one. For example, on his view, for each set theoretic universe V in the multiverse, there's a strictly larger one that includes all the sets in V and adds an extra subset of some set in V. Thus, Hamkins' view is ontologically realist but antirealist regarding the truth values for claims about mathematical objects.

In this paper, I'll note some questions (and a potential explanatory indispensability worry) that arise when we try to extend the multiverse understanding of set theory to applied mathematics. Most centrally, I'll explore an explanatory indispensability worry along the following lines. Certain widely accepted set theoretic/combinatorial explanations for physical facts use the assumption that there are precise facts about 'all possible subsets/ways of choosing' which give rise to lawlike, counterfactual-supporting constraints on how physical properties can apply to physical objects. If we reject this assumption (as the multiverse theorist must) a question arises about how to reinterpret or dispense with these scientific explanations?

Interestingly, unlike classic (explanatory) indispensability arguments (Quine, Baker and Colyvan) the explanatory indispensability worry I'll present attacks mathematical truth value antirealism, not mathematical object antirealism. The problem isn't that we seemingly need to *quantify over sets* to give certain kinds of intuitively good scientific explanations (Hamkins accepts the existence of sets and is happy to do that). Rather it's that we seemingly need to appeal to law-like constraints on all possible ways of choosing from some plurality of objects (and the idea that all possible ways of choosing can be witnessed in some hierarchy of sets) in order to best explain these phenomena.

In §2, I'll review some key facts about Hamkins' multiverse program, and its distinctive mathematical motivations. Then in §3 I'll raise a question about how to extend the multiverse proposal to a set theory with ur-elements, and a

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related explanatory indispensability worry. In §4 - §7 I'll consider some ways that a proponent of the multiverse could respond to it.

2 Hamkins' Multiverse

2.1 The Multiverse

In [5] Hamkins describes his multiverse proposal as a form of Platonism, which accepts the existence of many different set theoeretic hierarchies (with equal mathematical status) rather than one unique intended hierarchy of sets. On Hamkins' view certain set theoretic statements like the Continuum Hypothesis (i.e., the claim that there is no set intermediate in size between the real numbers and the natural numbers) are not true or false simpliciter, but merely true in some parts of the multiverse and false in others. In some universes in the multiverse CH is true and in others it is false, and there is no unique intended universe. Thus (as Hamkins vividly explains in the passage below) CH cannot be settled by finding intuitively compelling new axioms from which it can be proved or refuted. For mathematicians' experience reveals there are parts of the multiverse in which CH holds and parts in which $\neg CH$ holds.

> "[If some obviously true seeming mathematical axiom] ϕ were proved to imply CH, then we would not accept it as obviously true, since this would negate our experiences in the worlds having $\neg$ CH. The situation would be like having a purported 'obviously true' principle that implied that midtown Manhattan doesn't exist. But I know it exists. I live there. Please come visit! Similarly, both the CH and $\neg$CH worlds in which we have lived and worked seem perfectly legitimate and fully set-theoretic to us, and because of this, any [proof from ϕ that CH or that $\neg$CH] casts doubt on the naturality

of ϕ. [5]

Three further features of Hamkins' multiverse are worth noting here.

First, (I take it) Hamkins isn't proposing any kind of supervaluationist theory on which ordinary set theoretic claims are determinately iff true in every set theoretic universe and determinately false iff in every such universe (so the facts above show that CH is *indeterminate*). The idea is that set theorists study different set theoretic univereses in different contexts (as well as studying the relationships between them), like historians study different cities on earth. We don't say that it's indeterminate whether 'the city' has a population larger than 4 million, but rather by saying that there are many different cities, some of which have and others of which lack this property, and we must evaluate a historian's claim by determining which city they are talking about in a given context .

Second, Hamkins' proposal is inspired by a controversial interpretation of a mathematical technique called forcing. Hamkins suggests that for each set theoretic hierarchy V satisfying the ZFC axioms, we should accept that there is another (strictly wider) set theoretic hierarchy $V[G]$, the forcing extension of V. This expanded universe $V[G]$ adds a set G to V, where G is subset of a set (a partial order $\mathbb{P}$) that's already in V — along with other sets as needed for $V[G]$ to satisfy the ZFC axioms.

A version of this claim is uncontroversially true; if we work in some background notion of set theory we can prove that that every *countable model* of the ZFC axioms for set theory has a forcing extension (as mainstream/conventional approaches to forcing do)[2]. In contrast, Hamkins endorses the general claim that every set theoretic hierarchy satisfying the ZFC axioms has a forcing ex-

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[2]I omit mention of other technical devices Boolean valued models and inner models, for understanding forcing arguments without admitting that our background set theoretic universe could literally be widened in the way described above.

tension. He thus contradicts traditional/mainstream views that the intended hierarchy of sets already contains 'all possible' subsets all sets it contains, – so there can be no extended universe V[G], which adds a missing subset to a set (the partial order $\mathbb{P}$) this V already contains.

Finally, Hamkins asserts more powerful principles than the above claim about taking forcing extensions. Specifically he makes the provacative claim that, "Every universe V is ill-founded from the perspective of another, better universe."[4][3] (while no process of repeatedly taking forcing extensions can generate such a universe). Note that, in such cases, the natural numbers in V will be non-standard from the perspective of V', meaning that different set theoretic hierarchies can have different views about what number theoretic claims are true.

3 The Explanatory Indispensability Worry

Now to introduce my explanatory indispensability worry for Hamkins, note that we can seemingly use set theory to mathematically explain physical facts in ccertain cases. Here is a toy example (perhaps a more realistic example could be given using infinitely many galaxies with adjacency relations involving closeness to one another, and analogous explanations using more complex set theory also seem conceivable[4]).

Suppose we have an infinite map which isn't three colored (i.e., it's

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[3]To explain this talk of one set theoretic universe being well founded from the perspective of another, note that a model of ZFC set theory is well founded iff its ordinals are well founded. In particular, a larger model of ZFC set theory V' can see a smaller model of set theory inside it, V, as not being well founded, because V' may contain a subset of one of the ordinals o in V' that's missing from V' and which doesn't have an $\in$ least member.

[4]Similarly, it might be that reason why no one has succeeded in holding some (infinite) region on this map is that for every possible way of stationing defending troops in countries on the map (satisfying certain constraints) there's a way of stationing attacking troops such that... And perhaps we can cash this out more realistically by saying For each way of initially placing some FOO-particles in space, there's some way of placing BAR-particles such that, or some stable such and such will form...)

not the case that each map region is either red, green or blue, and no two adjacent map regions have the same color). From a naive point of view, a true and illuminating explanation for why the map isn't three colored might be that it's not three color*able*, in the sense that there is no set of a certain kind: the true hierarchy of sets with ur-elements doesn't contain any set coding a three coloring function

We seem to have something like a notion of 'all (logically/combinatorially) possible ways of choosing' objects, which has an important a priori relationship to both set theory and counterfactual-supporting constraints on how physical properties can apply to physical objects.

As regards physical objects, (we think) facts about 'all possible ways of choosing' can predict and explain regularities involving physical objects. If there's no logico-combinatorially possible way of choosing colors for map regions such that no two adjacent regions have the same color, we think two things follow. First the map isn't actually three colored. Second, the map couldn't 'easily' have been three colored: plausibly all very close possible worlds preserve the way that map regions are related by adjacency, so (because the same general logico-combinatorial constraints apply at all metaphysically possible worlds) it follows that the map isn't three colored at these close possible worlds either.

As regards set theory, traditional conceptions of the intended structure of the iterative hierarchy of sets seemingly apeal to this notion of all possible ways of choosing. Each layer of the iterative hierarchy of is supposed to contain sets corresponding to 'all (combinatorially) possible ways of choosing some sets that occur at lower levels[5] And for this reason it is supposed to contain sets corresponding to 'all possible' subsets of sets it contains.

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[5]That is, the hierarchy of sets supposed to have a structure which makes it logico-combinatorially impossible for some property to apply to some elements of a set, without there being a set that collects exactly those elements.

Because we have these background beliefs connecting set theory to counter-factual supporting constraints on physical properties and objects, we can use claims about what sets exist (e.g., there is no set coding a three coloring function for a given map) to evoke a counterfactual supporting explanation for a physical fact (that map is not three colored).

But what happens to this picture if we adopt Hamkins' multiverse understanding of set theory? As noted above, the multiverse theorist must deny that there's a 'full width' hierarchy of sets which contains 'all possible subsets' of sets it contains, as they think each universe V has an expanded universe V[G]. So they must either

- Reject the above notion of 'all possible ways of choosing' or

- Accept this notion but say that (for some reason) no single hierarchy of sets can contain all possible subsets of sets it contains.

And in either case we face question how to understand the set existence claims made in the toy physical explanations above. For example (following what Hamkins says about pure set theory) we can interpret the above explanations as talking about some particular set theoretic universe V. But if we do so (and say nothing else) we're not left with a genuine explanation. For the key background assumption that facts about set existence reflect counterfactual supporting logico-combinatorial constraints on how properties can apply must now be rejected. So (until we say more) there's no reason why the non-existence of a set coding a three coloring in some particular universe would even imply (much less explain) the fact that the map isn't actually three colored .

Thus, a problem analogous to classic Quinean-Putnam indispensability worries for mathematical nominalists arises. If we accept the multiverse theory, how are we to understand scientific theories which seemingly appeal to combinatorial constraints on 'all possible ways of choosing' (and a hierarchy of sets with ur-

elements that witnesses these facts)? Relatedly, we might want the multiverse theorist to account for intuitions that certain sentences (like the three-coloring and troop distribution hypotheses above) state *intelligible explanatory hypotheses* — whether or not these hypotheses are true. For one might hesitate to accept any philosophy of pure mathematics which required denying that the theories above express intelligible explanatory hypotheses.

Finally, note that the indispensability worry above arises regardless of whether there are universes in the multiverse which *disagree* about a set theoretic claim to explain a physical fact. Our problem is that Hamkins must reject the traditional bridge connecting set theory to lawlike constraints on how properties can apply to physical objects. And once we've done this, it is prima facie unclear if/why facts about what sets exist in the universe *or in the multiverse as a whole* should imply or explain constraints on the behavior of physical objects.

Now let's consider some possible answers to the above challenge.

4 A physically preferred V_p

The first strategy I want to consider mirrors the idea common idea that there's a physically correct geometry – a choice of geometrical axioms which (isn't mathematically special but) reflects the true structure of physical space and thereby certain lawlike (physically necessary) constraints on the behavior of physical objects.

The multiverse theorist could analogously say that there's a physically special set theoretic universe V_p, and propose a physical law which imposes necessary constraints on how physical properties can apply to physical objects. Specifically, we might say that it's a **physical law** that V_p contains all 'sufficiently physically definable' subsets of sets, as follows.

Physical Comprehension: If x is a set in V_p, then V_p also includes

all 'sufficiently physically definable'[6] subsets of x.

The multiverse theorist could then interpret appeals to the intended hierarchy of sets in physical theories and explanations[7] as talking about this physically preferred hierarchy V_p.

So, for example, rather than saying that a map isn't three colored because it's not three colorable (i.e., no suitable function exists in the unique intended hierarchy of sets) a multiverse theorist might say the following.

> The map isn't three colored because the physically preferred hierarchy of sets V_p (which contains sets corresponding to all ways it would be physically possible for some physically definable properties to apply to the physical objects that exist) doesn't contain any function three coloring it.

Taking this approach lets the multiverse theorist preserve the intuitive counterfactual-supporting force of the traditional explanation. They just replace appeal to logical-combinatorial laws (which would have to apply to mathematically and physically defined properties alike) with appeal to more limited physical laws.

However, this approach has potentially significant costs. For we might wonder 'how does physics control the outcomes of seemingly random independent events (e.g., coin tosses), to avoid 'realizing' (i.e., letting us use physical vocabulary to define) a missing subset? '

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[6] For example, it satisfies comprehension for all English formulas ϕ with parameters ranging over physical objects and sets in V_p, but not other sets in other universes in the multiverse.

[7] Note that such appeals to set theory/combinatorial possibility aren't limited explanations by appeal to set theoretic/combinatorial impossibility like the theory above, but also physical theories like the following. From a traditional point of view, we could assert a definite and elegant physical hypothesis by saying the following about some scenario where infinitely many different fundamentally (quantum mechanically?) random events – call them 'coinflips' were going to take place.

- **Independent Flips**: "These coin flips are random and independent; all (combinatorially) possible outcomes are physically possible."

  - i.e., For every set of coinflips x (in set theory with ur-elements[8]), it's physically possible that exactly the coinflips in x will turn up heads.

Maybe it's just a brute physical law that the outcomes of coinflips and painting countries etc. always avoid letting one define the missing subset. But this can be awkward. Is it plausible that a physical law steps in and stops the map from being three-colored in a way that corresponds to one of those missing subsets (and does the same to prevent three scenting, three texturing etc.)? Arguably the concept 'suitably physically definable' properties is too unnatural to figure in a plausible fundamental physical law.

5   Physical Comprehension Known by Stipulation/Charity

To avoid the above awkwardness, we could try to simply stipulate the Physical Comprehension principle above (i.e. stipulate that when using set theory in science we're always talking about background set theoretic universe which satisfies Physical Comprehension), while rejecting the claim that Physical Comprehension holds because of any physical or logico-combinatorial law.

A multiverse theorist of this kind can say 'Maybe the map won't ever be three colored because the background hierarchy of sets V (some set theroetic universe chosen to contain sets witnessing how all physical properties actually apply) doesn't contain a function three-coloring it.' But provide only a dormative virtue non-explanation. It's no better than saying, 'The reason why Jake doesn't have a driver's license is that the list of all people who hold a driver's license doesn't include him'. For example, neither pseudo-explanation preserves the intuitive counterfactual supporting force of our original three coloring explanation. They don't rule out the possibility that their explanandum is a complete fluke: that that map could very easily (at very close possible worlds) have been three colored or Jake could very easily have learned to drive.

6 Appeal to First Order Logic

A third response to our indispensability worry attempts to replace set existence claims with claims about first order logic in physical explanations like the three coloring and troop explanations mentioned above.

This approach is particularly appealing if we assume that all physical explanations the multiverse theorist needs to account for have a simple logical structure, similar to the three-coloring explanation in §**??** (perhaps because one only accepts the Quinean version of our explanatory indispensability challenge and thinks that more complex explantions are never used in actual science). For we can prove that, in each particular case where a map isn't three colorable, there will be some true non-mathematical, purely first order logical sentence about the countries which implies that if every country is either red, green or blue, two adjacent regions have the same color[9].

Thus, one might think the following. Whenever the traditional non-three colorability claim is true of some map, there's an alternative explanation which deduces the fact that the map isn't three colored from finitely many physical facts about which countries on the map are adjacent to each other and (robust, counterfactual supporting) first order logical laws. Thus we can replace all non-three colourability explanations we accept with simple deductions from physical facts and logical laws!

However, a dilemma arises when we try to cash this proposal out.

On one hand, the multiverse theorist *could* replace the traditional set theoretic non-three colorability explanation with something of the form below (where the ellipses are filled in with some specific first order logical deduction).

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[9]Consider an infinite theory in a language with names for every map region and sentences specifying all adjacency relations. By completeness for first order logic this theory is consistent iff contradiction can't be derived from it. If contradiction can be derived from it, contradiction can be derived from finitely many sentences within it (proofs can only use finitely many premises). Thus, there's a finite collection of sentences in this language which imply contradiction.

'That map will never be three colored because it contains countries
$A_1 \ldots A_n$ which are related like such-and-such and hence by ... the
map isn't three colored'

But this approach faces two problems. First, the explanations we get seem
significantly less unifying and explanatory than the original explanation. In [6]
Putnam famously contrasted the unifying high-level explanation 'this can't fit
through that because this is a square peg with side length ... and that is a
round hole with diameter...' with the microphysical explanation one might give
for the same fact. And demands to provide such unifying general explanations
are commonly cited in the conventional indispensability literature as challenges
for the nominalist[10]. Replacing general/higher level claims about non-three
colorability with a deduction of non-three coloring from some specific facts about
adjacency relations on this map involves the same intuitive loss of generality and
explanatory power.

Second, this strategy for dispensing with traditional set theory isn't suf-
ficiently widely applicable. For it seems, we might rationally entertain and
perhaps even know a claim like the traditional non-three colorability explana-
tion while *not knowing* any of the specific facts about adjacency that first order
logically entail the map isn't three colored. Perhaps we would need to know
such facts to prove that the map isn't three colorable. But, as recent work like
[1] notes, there are cases where we can explain physical facts by appeal to a
mathematical claim whose truth we rationally suspect but haven't proved[11].

To fix both problems, we could instead semantically ascend and talk about
derivability in our scientific explanation, rather giving a specific first-order log-

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[10]See, for example the criticisms Hartry Field's proposed infinitary nominalistic paraphrase
of Newtonian mechanics in [3] in works like [2].

[11]For example, Baker notes that scientists correctly hypothesized that bees had hexagonal
honeycombs because this allowed for an optimal of side length to area (under certain con-
straints). And he suggests that scientists could appeal to this mathematical fact to correctly
(if not exhaustively) explain why honeycombs were hexagonal before they had any proof of it
(i.e., at a time when this was only a plausible/motivated conjecture).

ical deduction. We might say:

> The map isn't ever three colored because it's FOL derivable from
> *some* true sentence about how finitely many countries are related by
> adjacency that it won't be three colored.

However this raises problems because of the connection between derivability
and the natural number structure together with the fact that Hamkins' Multi-
verse contains set theoretic universes that differ on well-foundedness, and hence
intended models of the natural numbers.

But if you don't think there's an intended natural number structure, it's not
clear how the above talk of derivability can be understood. We'd normally say
that a sentence is FOL derivable from some premises iff there is a number which
codes a corresponding proof. But Hamkins thinks that for each set theoretic
universe V there's another universe from whose perspective V has a nonstandard
copy of the numbers. And, arguably, unless I know I'm are talking about a
hierarchy with the true intended numbers, the claim that my background set
theoretic universe contains a 'number' coding a derivation of ψ from some truths
Γ, shouldn't rationally cause me to accept that ψ. Thus it's not clear that
Hamkins can accept this traditional notions of derivability, or how he would
formalize the derivability claim above.

Note: the claim here isn't that some set theoretic universes could contain
fake *three coloring functions*. Rather it's that some set theoretic universes will
contain numbers corresponding to fake *proofs* of non-three coloring from some
collection of truths about adjacency relations on the map. Given this fact,
there's a problem about whether anything the multiverse theorist can say (about
the numbers/sets in some universe) expresses the content we expect and need
the claim that it's FOL derivable from some true sentences about adjacency
relations on the map that the map isn't three colored, to express.

So, to summarize, we have the following situation. In fact, whenever a map isn't three colorable, we can (from a realist point of view) derive the fact that it won't be three colored from finitely many truths about adjacency relations the map in FOL. But it's not clear that there's any claim that one can make about Hamkins' multiverse that expresses this claim.

7   Appeal to the Whole Multiverse

The final response to our explanatory indispensability worry I want to consider replaces appeal to an intended set theoretic universe with claims about the multiverse as a whole. We might (although I'll suggest this is a rather unnatural reading) imagine Hamkins as thinking the following.

There are genuine laws of logical/combinatorial possibility. It's just that no single set theoretic universe could contain sets corresponding to all possible ways of choosing some of elements from a set that it contains. Rather, the multiverse as a whole witnesses the full range of combinatorial possibilities in the sense that, e.g., every combinatorially possible ways of choosing some objects from our physical ur-elements is witnessed by the existence of a set in some universe. In this case, facts about what sets exist somewhere in the multiverse reflect facts about logical possibility and thus get us at counterfactual supporting lawlike constraints on how all properties can apply. Thus, we can replace traditional claims about what sets exist in the intended hierarchy of sets V with claims about what sets exist anywhere in the multiverse. So we might rewrite our original non-three coloring explanation as follows:

> The map isn't three colorable because no universe anywhere in the multiverse contains a set three coloring it — and the multiverse contains sets witnessing all possible ways of choosing, so it follows from this that general combinatorial constraining on how any properties

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can relate any objects (together with the actual adjacency relations between countries on the map) rule it out.

This idea is particularly attractive if we narrow our attention to explainations with the simple ($\Pi 1$) structure of the non-three coloring example. For note that, unlike CH, facts about whether a three coloring exists can't be perpetually switched on and off by forcing. We can show that passing from V_p to a forcing extension V[G] can never change whether a map looks three colorable.

- Clearly if V_p has a three coloring then so does every expanded universe like V[G].

- On the other hand, we can show that if our V_p doesn't contain a three coloring function, taking forcing extensions will never add such a function.

Thus, if we could assume that all sets in the multiverse could be produced by some sequence of forcing extensions (!) from some base universe V_p satisfying Physical Comprehension, we could take claims about the non-existence of a three coloring function in V_p to implicitly tell us about the non-existence of a three coloring function anywhere in the multiverse (and thereby at general counterfactual supporting combinatorial constraints on the behavior of all objects).

Unfortunately however (as we saw) Hamkins explicitly denies the above premise that there's a single core universe from which all universes can be reached by repeatedly taking forcing extensions. So this argument that we can think about the non-existence of a three coloring in a universe V_p (stipulatively chosen to satisfy physical comprehension) as implicitly telling us there's no three coloring function anywhere in the multiverse fails.

One might instead cash out applied mathematics by explicitly talking about what sets exist anywhere in the multiverse (rather than by taking facts about

some single universe V_p to implicitly reveal such facts about the multiverse, as above). But this is unappealing. If we can quantify over all sets anywhere in the multiverse for the purposes of doing applied math, why can't we do so for the purposes of pure mathematics?

8   Conclusion

In this paper I've presented an explanatory indispensability worry for Hamkins' Multiverse approach to set theory. I then suggested a few different strategies for answering this worry, and noted some problems for each. In doing this, I don't claim to have refuted Hamkins' multiverse proposal. However, I do hope to have raised serious a question about how to understand applied mathematics, which the multiverse theorist must answer (and whose answer is likely to clarify important things about what they mean to say about pure mathematics as well).

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