“We now argue that the domain and range of R are initial segments of V, V' respectively. We note that if x is a set in V and x is in the domain of R then it’s possible (speaking loosely) that x in the domain (range) of g and g isomorphically maps some $V_0 \leq V$ to $V'_0 \leq V$ then, since x must be available at some ordinal u in V_0 (V'_0) it follows that x is available at some ordinal in the domain (range) of R. Similarly, if o is an ordinal in the domain (range) of R and $u < o$ then u is in the domain (range) of R. Thus, by *Lemma E.1 (Initial Segment)* we can infer that the domain of R is some initial segment $\hat{V} \leq V$ and the range is some initial segment $\hat{V}' \leq V'$ and $\hat{V} \cong_R \hat{V}'$.”

Context: You asked about the lemma which says that whenever we have V and V’ satisfying the characterization of initial segments one can be isomorphically mapped to an an initial segment of the other.

Before the quoted passage, we used modal comprehension to get the possibility that R applied so as to (in effect*) satisfy the following definition: R(x,y) is true iff it is possible for there to be $V_0 \leq V$, $V'_0 \leq V'$ and an isomorphism g of V0 to V0’ taking x in V0 to y in V0’.

*[I say `in effect’ because the above statement quantifies in. I officially don’t allow quantifying in as meaningful and simulate it using the trick in section \ref{}. Also the above sentence speaks about the possible existence of a function g, rather than the logical possibility that a relation N applies functionally]

In this passage,

1. I use the Initial Segment Lemma (E1) to argue that

-the domain* of R, $\hat{V}$, is an initial segment of V

*Explanation: When I talk about $\hat{V}$ as the domain of R here, I mean, by a slight abuse of notion, a $(\widehat{set}, \widehat{ord}, \hat{\epsilon}, \hat{\lesssim}, \widehat{@})$ where structure where $\widehat{set}, \widehat{ord}$, etc . are the restrictions of the set, ord etc. relations for V, to the domain of R

- the range of R, $\hat{V}'$, is an initial segment of V’

However, I went into way too much detail about how to check clauses needed for this lemma, proving things like: if, for some x at level u, there’s a y such that R(x.y), then for all x’ at level u, there’s a y’ such that R(x’,y’) .

2. I note that since R clearly maps from all its domain to all its range, we just need to show that R is functional and onto to get the conclusion that $\hat{V} \cong_R \hat{V}'$.

 I should have given more detail about this part.

Here’s more detail about proving that R is functional. The claim that it’s 1-1 is just the reverse.

Suppose, for contradiction, that R wasn't functional (i.e. we'd have some x in V related by R to distinct y and z in V'.)

By the fact that V is an satisfies our characterization of an iterative hierarchy structure, we can assume we can assume that x occurs at level u of the cumulative hierarchy and no counterexamples occur at earlier levels.

By our characterization of R, possibly there is a g satisfying the conditions in our characterization of R such that g(x)=z, and possibly there's a g' satisfying the conditions in our characterization of R such that g'(y)=z. And compossibility allows us to infer the possibility of a single scenario with* both g and g' .

*[That is, it's possible that some otherwise unused relations G and G' (simultaneously) apply so as to satisfy the descriptions of g and g' above respectively.]

To get contradiction, we then reason about this single possible situation as follows. We

- argue that u must be the first layer at which g and g' disagree (since it is the first layer at which R fails to be functional, hence the first at which any possible g and g' disagree)

-argue that, since g and g' agree on everything available below layer u (and hence on all elements of x, and ordinals below u) and are isomorphisms (respect ∈ and <) and map onto initial segments of V, they must agree on x.

For example, we can see that g'(x) is a subset of g(x) as follows. By the fact that g is an isomorphism, every element of g(x) is g(z) for some $z \in x$. Such a z must be available at a layer below u. So g'(z)=g(z). So we have both g(z) in g(x) and g(z)=g'(z) in g'(x). Similarly g'(x) is contained in g(x).