Arbitrariness Motivations for Potentialism

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Introduction

In [18] Soysal

- clarifies a worry about height that has been used to motivate (height) potentialism about set theory
    - plausibly rejecting certain simple responses as inadequate
- questions whether potentialists can better address this worry w/ a focus on
    - dependence potentialism (a la Linnebo, Studd and Parsons[14, 15, 13, 12, 20])
    - rather than minimalist potentialism (a la Putnam, Hellman and Berry[16, 10, 11, 1])
- proposes an alternative answer to the worry, which can seem puzzilingly similar to responses she rejects as inadequate.

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In this talk, I'll

- try to futher clarify and defend arbitrariness worries
- argue that (minimalist) potentialists can avoid such worries and address Soysal's criticisms
- suggest a way of using Soysal's more recent work on algorithmic conventionalist metasemantics to flesh out her sketched response to these access worries.

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Traditional actualist Platonists face a height challenge, which Soysal summarizes as follows:

- Why is the universe of sets not a set?

Some have tried to answer this worry by deriving contradiction from the assumption that there's a set of all sets

- e.g. from comprehension as per Russell's paradox
- just from foundation and pairing

But such answers seem unsatisfying,

- for they seem to answer the wrong version of 'Why is the universe of sets not a set?'
- we can distinguish two readings of this question...

- Why isn't there a universal set (set of all sets)?
  - c.f. Q: Why don't I have the property of being taller than myself? A: As a matter of logic and metaphysics, no one is taller than themselves.
  - Russell's paradox reasoning etc. may adequately answer this question
- Why does the hierarchy of sets stop where it actually does, rather than going up farther and containing a few more layers?
  - c.f. Q: Why am I not taller (e.g., why am I 1.6 meters rather than 1.7 meters tall?) A: It's a matter of genetics or nutrition...
  - this is question at issue with arbitrariness worries.

I would personally develop the challenge as follows (I'm not sure how much Z.S. agrees)

- Our naive conception of the intended height of the hierarchy of sets 'it goes all the way up, darn it' yields Buralli-Forti paradox.
- There's no widely-accepted coherent replacement conception of an intended height for the hierarchy of sets that purports to be categorical
    - as notions like 'all the way up', 'all possible ways of choosing', etc do
- So traditional (actualist) iterative hierarchy Platonist can seem committed to
    - a mysterious and unattractive stopping point, not determined by anything in our conception of mathematical objects
    - multiplying joints in nature beyond necessity

Burali Forti Details I

More specifically, naively it might be tempting to say the following

> ***Naive Height Principle:*** *If some objects are well-ordered by some relation* $<_R$*, there is an initial segment of the hierarchy of sets isomorphic to these objects under the relation* $<_R$

But this naive conception cannot be correct.

- For consider the way objects are well ordered by the relation $x <_R y$ iff
  - x and y are both layers in the hierarchy of sets and x is below y
  - or x is a layer in the hierarchy of sets and y is the Eiffel tower

Potentialism

Potentialist explications of set theory promise to avoid this arbitrariness problem by explicating set theoretic statements to replace

- apparent quantification over a single intended hierarchy of sets with some height
- with claims about how it would be (in some sense) possible for initial segments of the hierarchy of sets (or objects with the intended structure thereof) to be extended.

They thereby avoid commitment to a favored stopping point for the hierarchy of sets (or at least to one that's relevant to mathematics).

Let me now say a little about

- minimalist potentialism (the kind of potentialism I aim to defend in this talk)
- and how it differs from dependence potentialism

Minimalist potentialism grows from Putnam's explication of set theory as considering what 'models' of set theory are, in some sense, *possible* and how such models can be extended. In [17] he

- considers 'standard models' of set theory built of concrete objects, e.g., pencil dots that are related by physical arrows.
- understands set-theoretic statements as claims about what such concrete models are possible, and how they can be expanded.

Hellman sharpens and develops this proposal by understanding

- the relevant notion of possibility $\Diamond$ as logical possibility
  - approximately interdefinable with entailment (e.g., $\Diamond\phi$ iff $\neg\phi$ is not entailed by empty premises)
  - something we have independent reason to take as primitive rather than cashing out using set theory [8, 9, 2, 6])
- 'standard models' as models which (basically) satisfy ZFC_2.

However versions of minimalist potentialism have using different machinery have been proposed.

Notably, all versions of minimalist potentialism eliminate talk of sets and elementhood, replacing it with e.g.,

- second-order quantification 'It's logically necessary that ($\forall X, f$ if $ZFC_2[set/X, \in /f]$ then...)'
- non-mathematical one and two place relations 'It's logically necessary that if the ink dots and arrows satisfy ZFC_2 then..'

For example, Hellman's minimalist paraphrase of "$(\forall x)(\exists y)(x \in y)$" looks like

$$\Box(\forall V_1)(\forall x)[x \in V_1 \to \Diamond(\exists V_2)(\exists y)(y \in V_2 \wedge V_2 \geq V_1, \wedge x \in y)]$$

(where quantification over all V_i is shorthand for quantification over all second-order objects X, f satisfying some axioms like ZFC_2)

Conditional Logical Possibility, Very Briefly

My favored version of minimalist potentialism appeals to an enhanced logical possibility operator $\Diamond_{...}$, that lets us talk about what's logically possible holding fixed (structural facts about) how some relations $R_1 \ldots R_n$ apply.

To quickly motivate this notion, consider the following example:

If a physical map is not three-colorable we might say:

$\neg\Diamond_{adjacent,country}$ [Every country is either yellow, green or blue and no two adjacent countries are the same color]

'It's logically impossible, given the (structural) facts about how 'is adjacent to' and 'is a country' apply on the map above, that every country is either yellow, green or blue and no two adjacent countries are the same color.'

Using this $\Diamond_{\dots}$, we can rewrite Hellman's minimalist potentialist paraphrases of set theory to remove

- second order quantification, plural quantification
- all quantifying in to the $\Diamond$ of logical possibility.

Some Advantages:

- can capture intuitive content of extendability claims relevant to potentialist set theory without taking controversial positions on de re modality (what is possible for objects vs. what is possible while preserving structural facts about how relations apply)
- fits with structuralist intuitions

Dependence theoretic potentialism

In contrast, dependence theoretic potentialists

- acknowledge the existence of special objects called ‘sets’
- but interpret set theory potentialistically, as talking about what sets **could be formed** (where the fact that these are sets plays an essential role not captured in mere axioms used in potentialist paraphrase)
- In what sense could more pure sets be formed? Dependence potentialists have (variously)
  - left this a sui generis notion
  - appealed to a notion of intepretational possibility

Dependence potentialists don't specify the intended structure of an iterative hierarchy in their paraphrases.

- Instead, they take many facts about the kind of structure of sets you can start talking in terms of to fall out of the *interpretational essence* of sethood and elementhood (or similar)
  - e.g. extensionality is preserved in all relevant reinterpretations of 'set' and 'element'

This lets dependence theorists give shorter logical regimentations for set theory than minimalists can.

Minimalist paraphrase of "$(\forall x)(\exists y)(x \in y)$" (where quantification over all V_i is shorthand for quantification over all second-order objects X, f satisfying some axioms like PA_2)

$$\Box(\forall V_1)(\forall x)[x \in V_1 \to \Diamond(\exists V_2)(\exists y)(y \in V_2 \wedge V_2 \geq V_1, \wedge x \in y)]$$

Dependence paraphrase (using 'set' as a primitive):

$$\Box(\forall x)[set(x) \to \Diamond(\exists y)(set(y) \wedge x \in y)]$$

So I admit one practical advantage for the dependence theorist: shorter and cleaner looking paraphrases!

- But typesetting isn't all (and minimalists could Linnebo's notation as an abbreviation).

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Now let's turn to Soysal's criticisms of potentialism as an answer to the arbitrariness worries above.

- She focuses on dependence potentialists
- I'll argue minimalist potentialists can answer these criticisms.

To set up her criticisms, Soysal suggests there are deep similarities between potentialist and actualist understandings of set theories, via facts like the following:

- There's an isomorphism between
  - possible worlds w_α in a Kripke model for the potentialist's modal notion $\Diamond$
  - the stages V_α, in a cumulative hierarchy of sets built up by taking powersets V_0, V_1 etc.
- Potentialism and actualism are mutually interpretable. While preserving the truth of all popular axioms you can
  - interpret potentialist claims as quantifying over stages in an actual hierarchy
  - (as well as) interpreting ordinary, seemingly actualist set theory potentialistically

She also notes that dependence potentialists are sometimes willing to talk (at least loosely) about

- potentialists (merely) having “powerful instruments for studying the same subject matter under a finer resolution” (Linnebo)
  - c.f. Putnam circa [16] regarding modal and non-modal perspectives on math as equally legitimate
- a “potential hierarchy of sets" as a completed totality, and quantify over all possible sets.

For example, Linnebo seems to allow talk of a "potential hierarchy of sets" as a completed totality, when he motivates the reflection principle $\phi^{\Diamond} \rightarrow \Diamond\phi$ (where $\phi^{\Diamond}$ is the potentialist explication of ϕ), as follows

> *The truth of a claim in ‘the model’ provided by the potential hierarchy of sets ensures that the claim is possible. For a claim ϕ to be true in this ‘model’ is for ϕ to be true when all its quantifiers are understood as ranging over all possible sets, including ones not yet formed. But for ϕ to be true when understood in this way is simply for its potentialist*
> ‘ *translation $\phi^{\Diamond}$ to be true*

Soysal uses these criticisms to raise a few interesting worries as follows

> *The potential and iterative hierarchies are isomorphic, and modal and non-modal set theories are mutually interpretable. This means we cannot get any grip on the potentialist's modality by merely considering these sets of true sentences containing '$\Diamond$' and '$\Box$'. If, moreover, we are given no independent grip on potentialist's notion of modality (because we are told it is primitive and idiosyncratic to set theory), then what stops us from simply interpreting the domains of the worlds w_α as stages V_α defined in ZFC? What exactly is added by the '$\Diamond$' and '$\Box$' in front of quantifiers? Potentialism on this option starts to look like a notational variant of set theory. And this surely affects its explanatory power: To say that the universe of sets is not a set because it is "potential" in that at any stage, we "can" form more sets in this unspecified and idiosyncratic sense of "can" is not far from giving a dormative virtue explanation, or saying nothing at all.[18]*

Specifically I take Soysal to be suggesting that if "The potential and iterative [actual] hierarchies are isomorphic" etc. this raises worries about:

- Reference:
  - what stops us from interpreting potentialist modality as referring to initial segment of an actual structure ?
  - we can't grasp the meaning of $\Box$ and $\Diamond$ just by considering truthvalues for sentences/axioms
- Explanatory power: appeals to a sui generis under-motivated modal notion seemingly can't do much explanatory work.
- Difference making: Given the above great structural similarity between potentialism and actualism, it's implausible that going from one to the other could help with arbitrariness worries.

‘Potential and Iterative Hierarchies are Isomorphic?’

In response I will

- argue that a minimalist potentialist can and should resist
  - talk of a "potential hierarchy" of sets
  - the arguments for deep structural similarities between potentialist and actualist understandings of set theory above
- address the specific worries (about reference, explantory power and difference making) above

I claim that potentialists should reject talk of “a potentialist hierarchy"

- Giving potentialist explications for first order set theoretic sentences does require accepting the legitimacy of such talk
  - or the meaningfulness of second order or plural quantificaiton over all sets, or even de re claims about ‘what sets could exist’/‘what objects are possible’
- And the Burali-Forti ideas motivating potentialism above suggest
  - any actual hierarchy must fall short of going ‘all the way up darn it’ (as per conceptions of the height of the hierarchy of sets)
  - end at some seemingly arbitrary stopping point
- So no actual structure could provide an intended interpretation of set talk as the potentialist understands it.

Similarly, the potentialist should assign little importance to the existence of an isomorphism between

- Kripke models for the potentialist's modal notion (under the relation 'extends')
- initial segments V_α in an iterative hierarchy (under the relation 'extends')

For, I claim, the potentialist should regard both sides of the isomorphism as providing only (deeply) unintended interpretations.

- (for the reasons noted above) any actual hierarchy or kripke model must impose limits on size/height that intuitively don't apply to logical (/interpretational?) possibility.

Now let’s turn to Soysal’s specific challenges

Reference Challenge

Reference worry:“we cannot get any grip on the potentialist’s modality by merely considering these sets of true sentences containing ‘$\Diamond$’ and ‘$\Box$”’

- Plausibly deploying inference rules for modal logic provides *some* of our grip on relevant notions
- But I grant that I’m taking modal vocabulary to have a meaning which transcends this, and indeed all facts about truthvalue of sentences
  - the considerations above suggest interpretations of $\Diamond$ claims as asserting the existence of models/initial segments of an actualist hierarchy of sets are
    - unintended
    - overly restrictive/narrow (even if they get all truth values for sentences right, and vindicate our inference methods)

Reference Challenge

Admittedly you can still ask : how can we refer to logical possibility rather than these alternatives?

But this seems like neither more or less of a problem than accounting for reference to

- physical possibility (understood as transcending facts about the actual pattern of events) or metaphysical possibility, as oppose to variant notions that allow more/fewer things in a way that makes no difference to truthvalue
- objects like cats, rather than any interpretation of the extension of our physical predicates which makes the truthvalue of all sentences in our language come out right.

Ad feminam, I might add:

- as friend of elaborated possible worlds semantics, wouldn't Soysal want to say that we can refer to a definite notion of metaphysical possibility somehow?
- How are prospects for referring to a definite favored notion of metaphysical possibility any worse?

Next Soysal raises a worry about

Explanatory power worry: appeals to a sui generis under-motivated notion seemingly do much explanatory work.

> *"To say that the universe of sets is not a set because it is 'potential' in that at any stage, we 'can' form more sets in this unspecified and idiosyncratic sense of 'can' is not far from giving a dormative virtue explanation, or saying nothing at all"*

Perhaps this identifies a problem for dependence potentialists like Linnebo and Studd who

- claim there are (or could be) sets
- and invoke an idiosyncratic notion like interpretational possibility (distinct from logical and metaphysical possibility) to make sense of this.

However I think minimalist potentialists can avoid trouble as the notion of logical possibility is fairly strongly independently motivated...

Why accept $\Diamond$ as primitive logical vocabulary (rather than analyzing it in terms of the existence of set models)?

- Intuitively, if ϕ is logically necessary it must be true in the actual world. [8, 9, 2, 5, 7]
  - But the mere absence of set counter models (which might, e.g., have certain limits of size not relevant to the whole of reality) doesn't generally or clearly ensure this.
  - though the completeness theorem happens to ensure this when ϕ is in FOL.
- (Boolos) “one really should not lose the sense that it is somewhat peculiar that if G is a logical truth, then the statement that G is a logical truth does not count as a logical truth, but only as a set-theoretical truth"[2].

Difference Making Worry

Difference making: Given the structural similarity between potentialism and actualism , how can going from one to the other address arbitrariness worries?

Minimalist Potentialists automatically avoid one kind of arbitrariness worry for dependence theorists

- They don't have to answer 'but how many sets are there actually?', as their paraphrases eliminate 'set' and 'element'
- So (unlike dependence theorists) they needn't take current set theoretic practice to (somehow) fix a height for an actual hierarchy of sets

But they do still face an arbitrariness question as follows...

A Modal Version of Arbitrariness Worries?

> *"If we ask why the actual hierarchy stops at a certain point, rather than going higher we can ask a similar thing about the modal notion used in potentialism — why does it allow these things and not more?"*

However, I claim 'why aren't more things logically possible?' is much less troubling than 'why doesn't the hierarchy go up further?' for the following reason:

- In the case of logical possibility we both
  - grasp a seemingly cogent and precise/joint carving notion
  - whose coherence we have no positive reason to doubt (unlike naive conceptions of the height of the hierarchy)

In contrast, we have no such conception of the intended height of the hierarchy of sets

- the naive conception of the sets going ‘all the way up’ must be rejected as incoherent, as per Burali-Forti paradox
    - for any actual hierarchy, we can define a longer well ordering only using actual objects
    - and the idea that it would be logically possible to have a larger hierarchy is intuitive and (and gets some support from mathemtical practice via talk of classes)
- there is no widely accepted replacement conception of a unique intended height for the hierarchy of sets.

Admittedly, we can easily state restrictive conceptions of height e.g.,

- V is the smallest model of ZFC_2,
- V is the shortest intended width hierarchy that matches potentialist truth-values for all first order set theoretic statements

But such conceptions tend to be rejected as too small

- (potentialist interpretations have the advantage of allowing arbitrarily large possible structures to be relevant to set theory)

Accordingly, actualists seem to face a question about height that's not analogous to (and requires more of an answer than) 'why aren't more things logically possible?'

- why does the actual hierarchy of sets stop where it does, given
  - our naive conception of the intended height of the hierarchy ('all the way up darn it')turned out to be incoherent/paradoxical,
  - nothing in our remaining conception even seems to pick out a unique height

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Soysal proposes the following alternative response to height challenges for traditional actualist set theory.

> ***Conception-Based Explanation*** *The universe of sets is not a set because the supposition that it is contradicts some axioms of ZFC, and these axioms are part of the iterative conception of sets.*

"What the conception-based explanation adds to the minimal explanation [provided by citing Russell's paradox or the like] is the claim that the axioms that prohibit the existence of a universal set are part of the iterative conception of sets."

But this can still seem unsatisfyingly similar to the earlier answer to arbitrariness worries, which Soysal (I think rightly) rejects.

- For it can still seem to answer the wrong version of ‘why isn’t there a set containing all (actual) sets?’
- by providing no plausible account for why an actualist hierarchy might stop at any particular point.

However, I'll suggest that Soysal's recent work on algorithmic conventionalism [19] suggests a way of fleshing out the above remarks that would more distinctively and satisfyingly addresses height arbitrariness worries (as I understand them).

In [19], Soysal proposes a rather austere metasemantic view on which (to somewhat simplify) the meaning of a sentence is given by a combination of

- a set of possible world (at which this sentence is true)
- algorithms that that let us do things like recognize semantic facts about which sentences express truths.

Accordingly, we might read Soysal as proposing an answer to arbitrariness worries for actualism that

- resembles the potentialist response defended above
- but is far more radical, in the following sense...

The minimalist potentialist story defended above avoids height arbitrariness worries by holding that

- our words ‘set’ and ‘element’ don’t refer to any favored actualist hierarchy
- rather, we lock on to a notion of logical possibility, which helps systematically determine truth values for first order set theoretic sentences (as per minimalist potentialist regimentations).

In contrast, perhaps Soysal avoids height arbitrariness by saying

- Our mathematical practices don't secure reference to either
  - a favored actualist hierarchy with some height **OR**
  - a favored modal notion of logical possibility (or interpretational possibility etc.)
- Rather mathematical sentences are just more directly associated with
  - a set of possible worlds
  - an algorithm for recognizing linguistic facts about which sentences express truths.

My main concern about this imagined (Soysal's?) proposal is just the usual one for mathematical conventionalism/formalism:

- Does this story allow room for unprovable truths about the natural numbers? Provability?
- If not, how does it resist familiar arguments from Gödel incompleteness to proof-transcendent facts about the numbers/provability?

Perhaps there's also a disagreement about philosophical method lurking:

- I've been taking appearances of reference to a unique joint carving notion (e.g., of logical possibility) at face value, until given a positive reason for doubt
  - as Burali-Forti worries do for naive height, and Russell's paradox for naive conceptions of the sets
- In contrast, Soysal might say her proposal should be favored as
  - leaving fewer hostages to fortune re: whether we succeed in referring to a unique coherent notion of logical possibility (or the like)
  - extends attractive possible world semantics to the mathematical case
  - removes the apparent oddness/parochialness of saying names in my language refer, while some names in a quantifier variant language that adds talk of incars don't.

However, maybe one doesn't have to chose and Carnapian tolerance is appropriate.

- c.f. Chihara's point[3] that sometimes have two different logical regimentations of the same thing -normal and nonstandard analysis
- Eswaran's paper[4] on axioms as an overlapping consensus, that lets mathematicians avoid argument over philosophical
- Perhaps something like Soysal's proposal provides a valuable fallback interpretation (should apparent grasp of logical possibility turn out to be an illusion)

Conclusion

In this talk I have tried to

- sharpen/articulate arbitrary height worries for actualist set theory
- defend minimalist potentialism as a response from Soysal's objections
- (speculatively!) flesh out Soysal's favored response by connecting it to some of her later work

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