CHALMERS, QUANTIFIER VARIANCE AND MATHEMATICIANS' FREEDOM

SHARON BERRY

1. Introduction

Philosophers of mathematics have been much struck by mathematicians' apparent freedom to introduce new kinds of mathematical objects, such as complex numbers, sets and the objects and arrows of category theory. For example, in a recent *Australasian Journal of Philosophy* paper Julian Cole writes, "Reflecting on my experiences as a research mathematician, three things stand out. First, the frequency and intellectual ease with which I endorsed existential pure mathematical statements and referred to mathematical entities. Second, the freedom I felt I had to introduce a new mathematical theory whose variables ranged over any mathematical entities I wished, provided it served a legitimate mathematical purpose. And third, the authority I felt I had to engage in both types of activities. Most mathematicians will recognize these features of their everyday mathematical lives."[6].

In this paper, I will explore a way of using recent work on quantifier variance to explain this apparent freedom to introduce theories about new kinds of mathematical objects. In [5], David Chalmers suggests a way of describing a class of alternative quantifier senses which are more ontologically profligate than our own using appeals to set theoretic models. I will suggest a modification of this proposal which frees it of certain arbitrary limitations on size by replacing appeals to set theory with appeals to an (independently motivated) notion of broadly logical possibility. Once amended in this way, Chalmers' technique allows us to flesh out a Neo-Carnapian explanation for mathematicians' freedom to introduce new kinds of mathematical objects which avoids some major problems for existing accounts.

2. Motivating the Project: Existing Approaches

When attempting to explain mathematicians' freedom to introduce new kinds of mathematical objects, we want a theory which

(1) avoids ruling out intuitively acceptable mathematical practices
(2) captures the metaphysical necessity of mathematical truths and
(3) honors the apparent similarity between mathematical and other ordinary language existence claims, e.g., the similarity in inferential role between 'there are numbers' and 'there are cats'.

However, many existing approaches have trouble honoring one or more of the three desiderata above.

Limitative approaches like classic Set Theoretic Foundationalism, interpret mathematicians as talking about a fixed but large universe of mathematical objects. They explain mathematicians' freedom to stipulate by saying that mathematical objects are plentiful and diverse but reject the claim that objects corresponding to all coherent stipulations exist. Instead,they posit generous 'limits of abstraction' such that all *acceptable* characterizations of mathematical structures can be understood as truely describing (portions of) a single mathematical universe. Thus, for example, in the case of standard set theoretic foundationalism, acceptable stipulations will be those which have a 'standard' model in the hierarchy of sets. They then argue that all such acceptable mathematical stipulations will express truths because they truly describe suitable portions of a plentiful mathematical universe.

Limitative approaches have trouble satisfying the first desiderata: their choice of *which* limits to impose can seem unmotivated. For example, in the case of standard set theoretic foundationalism, this worry takes the following form: if the hierarchy of sets has some definite height, why doesn't the mathematical structure one would get by adding a layer of classes to this hierarchy constitute an acceptable object for mathematical investigation[1]?

To see how this problem arises more generally, note that one cannot say that the mathematical universe is rich enough to make all coherent hypotheses characterizing putative new types of mathematical objects true. For, as Boolos[3] and Uzquiano[15] have emphasized, not all coherent hypotheses about abstracta are compatible with one another. For example, although it seems coherent to say that every object belongs to a set, and it seems coherent to say that every plurality of objects has a mereological fusion, Uziquiano shows that someone who accepts certain natural first order axioms of applied set theory and a plausible list of first order truths about meriology cannot *conjoin* both claims on pain of logical contradiction. In essence, the problem is that Uziquiano's principles of applied set theory and mereology both include claims about how their respective objects relate to *all other objects* which imply incompatible consequences about the size of the universe as a whole[2].

[1]Combining set theoretic foundationalism with a **potentialist** approach to the hierarchy of sets (which denies that the hierarchy of sets has any definite structure and cashes out mathematical existence claims in terms of logically possible extendability) [8][11] would let one resist the this thought that whatever structure the hierarchy of sets has, it would be coherent for there to be a larger structure. This approach faces problems with the second desiderata like those which beset the hypotheticalist approach described below, insofar as as it reduces all mathematical object existence claims to claims about foundational objects (e.g. claims about set theory), which it says do not function like ordinary object claims and need to be specially paraphrased.

[2]For example, Uziquiano notes a conflict between "Atomicity: There are no objects whose parts all have further proper parts.", "Limitation of Size: Some objects form a set if and only if there is no 1-1 map from the entire universe into them." and commonplace axioms of set theory and mereology. Combining Atomicitiy with standard principles of mereology requires the universe as a whole to have size 2α for some cardinal α. But combining

One also cannot say that all coherent patterns of relationships between objects are realized in some *portion* of the mathematical universe – so that all coherent characterizations will be true with respect to some restricted sense of the quantifier. For intuitively we can make sense of the notion of all possible ways of choosing objects out of a collection. Given any structure which we take the mathematical universe to have, it appears that it would be possible and coherent for there to be a different structure corresponding to what you would get by adding a layer of classes to the original universe (i.e., adding a layer of objects which 'witnesses' all possible ways of choosing from the original objects). By Cantor's diagonal argument, this structure would be strictly larger than the original one. Thus it appears there is a way some abstract objects could be related to one another which requires the existence of too many objects to be realized by any portion of the total mathematical universe.

Institutional/Social Constructive approaches like Cole's[6] take mathematical objects to be 'institutional' facets of reality which "exist in virtue of collective agreement" and are, in some sense, created and "sustained in existence by a relevant group of people collectively recognizing or accepting their existence."[6] Just as lawyers can bring companies into being, so too mathematicians can ensure the existence of some suitable collection of mathematical objects just by choosing to accept certain existential claims about such objects. These views face problems with the second desideratum. Taking mathematical objects existence to be grounded in social facts in the same way that the existence of money or countries strikes many people as odd or counter-intuitive. It is also prima facie difficult to square the

Limitiation with standard axioms of set theory (which imply that *the sets* do not have size 2α for any α) yields the result that the universe also cannot have size 2α.

Institutional account with the idea that mathematical statements can be timelessly and necessarily true[3].

Hypotheticalist approaches hold that the true logical form of a mathematical utterance 'ϕ' is something like a conditional claim 'if D then ϕ', where D combines all the descriptions of intended structures of mathematical objects currently in play. They face problems with the third desideratum above: taking mathematical existence claims to have such a different logical structure and semantics and from existence claims about ordinary and scientific objects can seem ad hoc and (ceterus paribus) unattractive.

As Stanford Encyclopedia puts it, "the language of mathematics strongly appears to have the same semantic structure as ordinary non-mathematical language... the following two sentences appear to have the same simple semantic structure of a predicate being ascribed to a subject:

(4) Evelyn is prim.

(5) Eleven is prime.

This appearance is also borne out by the standard semantic analyses proposed by linguists and semanticists."[4]. This calls into question whether or not the hypotheticalist's conditionals can be seen as giving literal meaning of mathematical existence statements.

In the next section I will suggest that recent work on Quantifier Variance helps us articulate and develop an alternative, **Neo-Carnapian**, approach which avoids the problems above.

[3]Some proponents of Institutional approaches have attempted to address this worry by noting that standard acts of social construction (such as founding a company or granting an individual some important social status) can take effect retroactively. For example, Cole notes that sports authorities can retroactively rule that a player has been on the 'injured list' for the past two days, and he suggests that mathematical authorities can similarly rule that sets and numbers exist timelessly and amodally.

[4]This example comes from [4] pg. 288 via [10], but the point goes back to [1]

3. A Neo-Carnapian Approach

3.1. **The Strategy.** Characterizations of Quantifier Variance in the metaontology literature frequently combine two elements. First, they include a *multiplicity claim* to the effect that the "∃" symbol can take on a range of different meanings which are all (somehow) existential-quantifier-like. One might articulate this aspect of Quantifier Variance (Quantifier Variance$_M$) as follows[5].

> Quantifier Variance$_M$: The English word 'exists' takes on a range of meanings[6] in different contexts, such that
>
> - all these variant meanings satisfy the usual syntactic inference rules associated with the existential quantifier[7]
> - it is not the case that all these variant meanings must be understood as quantifier restrictions of a fundamental most generous sense of the quantifier[8].

Second they include a *parity claim*, Quantifier Variance$_P$, to the effect that all these meanings are (somehow) metaphysically on par. Thus, for example, Chalmers characterizes Quantifier Variance as (roughly) the idea that, "there are many candidate meanings for the existential quantifier (or

[5]This definition is heavily influenced by Sider e.g. [14]

[6]For ease of exposition I will sometimes talk about shifts in mathematical context as giving rise to shifts in the sense or meaning of the quantifier. However I don't mean to rule out the attractive idea that (strictly speaking) the meaning of the quantifier symbol always stays the same, and what changes is its contribution to the truth conditions for sentences - just as one might say that the meaning of 'now' remains fixed, but the way that utterances of 'now' contribute to the truth conditions for utterances shifts depending on the context of utterance.

[7]Specifically "($\exists I$) If $\Gamma \vdash \theta$, then $\Gamma \vdash \exists v\theta'$, where θ' is obtained from θ by substituting the variable v for zero or more occurrences of a term t, provided that (1) if t is a variable, then all of the replaced occurrences of t are free in θ, and (2) all of the substituted occurrences of v are free in θ." and "(∃E) If $\Gamma_1 \vdash \exists v\theta$ and $\Gamma_2, \theta \vdash \phi$, then $\Gamma_1, \Gamma_2 \vdash \phi$, provided that v does not occur free in θ, nor in any member of Γ_2."[12]

[8]The purpose of this second clause is simply to distinguish the kind of multiplicity required by Quantifier Variance from the bland claim that we sometimes speak with implicit quantifier restrictions, as when we say 'all the beers are in the fridge'.

for quantifiers that behave like the existential quantifier in different communities), with none of them being objectively preferred to the other."

I will argue that Quantifier Variance$_M$ can be used to give an attractive Neo-Carnapian explanation of mathematicians' freedom – whether or not one accepts any further parity claim along the lines of Quantifier Variance$_P$. Philosophers on both sides of recent debates about the defectiveness of ontology have been motivated to accept Quantifier Variance$_M$. In particular, they have been inclined allow non-metaphysicians substantial freedom to deploy variant senses of the quantifier which are not mere quantifier restriction of some maximally natural fundamental notion of existence (either because there is no such maximally natural quantifier sense[9] or because plumbers and scientists don't attempt to use it[10]). However, merely allowing *this* role for Quantifier Variance allows us to flesh out a Neo-Carnapian explanation of mathematicians' freedom along the following lines.

> **Neo-Carnapian explanation of mathematicians' freedom**: When mathematicians (or scientists or sociologists) introduce coherent hypotheses characterizing new types of

[9]Foes of ontology like Hirsch[9] maintain that we cannot succeed in giving the quantifier a special maximally natural meaning in the metaphysics room which comes apart from ordinary practice, and that in cases where ordinary practice is undecided between different variant senses of the quantifier there there may be no right answer to existence questions. Accordingly they invoke quantifier variance (between different acceptable precisifications of language uses on the street) to explain how ontological discussions about whether certain kinds of objects exist can be defective. They combine quantifier variance with either a parity claim that all different senses of the quantifier are on par, or rejection of appeals to the relative naturalness of quantifiers as meaningless.

[10]Friends of ontology like Sider [14] have used quantifier variance (between the street and the philosophy room) to capture the intuition that ordinary speakers non-philosophical utterances like 'there's a hole in a sink' can express uncontroversially true statements, despite the fact that there's a deep open question about what exists in the more fundamental sense relevant to the metaphysics room. They say that there is a unique maximally natural sense of the quantifier which ontologists aim to employ, and that it is a deep open question whether holes exist in this sense. However, they allow that there is also a different (perhaps less than maximally ontologically insightful) sense which the quantifier can take on in the context of ordinary life/plumbing discussions, on which sentences like 'There is a hole in this pipe.' can uncontroversially express truths.

objects, this choice behaves like an act of stipulative definition[11], which not only gives meaning to newly coined predicate symbols and names but can change the of meaning expressions like "there is", in such a way as to ensure the truth of the relevant hypothesis.

Thus, for example, mathematicians' acceptance of existence assertions about complex numbers might change the meaning of our quantifiers so as to make the sentence "there is a number which is the square root of -1" go from expressing a falsehood to expressing a truth. Similarly, sociologists' acceptance of ontologically inflationary conditionals like "Whenever there are people who... there is an ethnic group which ..." can change the meaning of our quantifiers so as to ensure that these conditionals will express truths.

3.2. **Advantages.** If this Neo-Carnapian explanation can be fleshed out in a plausible way (as I shall attempt to do in what follows), adopting it would let us satisfy all the desiderata mentioned above.

The explanation above avoids commitment to an apparently unmotivated distinction between acceptable and unacceptable posits. For the it lets us to say that all posits describing coherent mathematical structures succeed in changing the meaning of one's quantifiers in such a way as to ensure their own truth – even if it is not the case that all such posits can be understood as describing portions of a single large universe.

[11]Admittedly there can be some vagueness and arbitrariness about whether, for example, we act more like we *stipulated* that the numbers satisfy induction and then *derived* the least number principle or like we stipulated that the numbers satisfy the least number principle and then derived that they satisfy induction. Nonetheless showing how an explicit process of stipulation and deduction could produce just the kind of easy a priori access to existence claims about new mathematical objects which mathematicians seem to have has substantial power to dispel the apparent mystery of this easy access – even if one acknowledges the actual story is a little messier.

It also honors our desire for a uniform account of the meaning and logical form of existence claims about mathematical objects and grammatically similar existence claims involving scientific and ordinary objects like holes, shadows, electrons etc. For it allows us to say that a single notion of existence is relevant to different ordinary language existence claims "Evelyn is prim." and "Eleven is prime." in any given context, though future choices to start talking in terms of new kinds of objects (be they sociological objects like countries or literary objects like genres or mathematical objects) may change which notion of existence this is[12] [13].

Finally, it lets us avoid institutional approaches' unintuitive claims about mathematical objects' existence being somehow grounded in facts about human societies. For the neo-Carnapian makes no claim that mathematicians' acts of stipulation somehow create or sustain mathematical objects in existence. Rather they say that these acts of mathematical stipulations

[12]Relatedly, one should note that that using quantifier variance does not require one to accept that normal English employs verbally different expressions corresponding to at least two different quantifier senses (a metaphysically natural and demanding one and a laxer one), so that it might be true to say things bad-sounding things like "composite objects exist but they do not really exist" in certain contexts. With regard to any particular context we can fully agree with David Lewis that, "The several idioms of what we call 'existential' quantification are entirely synonymous and interchangeable. It does not matter whether you say 'some things are donkeys' or 'there are donkeys' or 'donkeys exist'...whether true or whether false all three statements stand or fall together."
.

[13]One might worry that combining this neo-Carnapian explanation of mathematicians' freedom with a realist approach to ontology creates pressure to think that mathematical existence assertions are (somehow) ontologically second class citizens in way that would conflict the uniformity intuitions noted above. However one should note that combining a neo-carnapian explanation of mathematicains freedom with Siderian realism about ontology only suggests that mathematical objects are ontological second class citizens in the same way that it *also* suggests holes and countries are ontological second class citizens. Easy access to existence claims in mathematics comes out to be a limiting case of easy access to ontologically inflationary conditionals like 'if physical facts are like this then there are holes...' 'if people are doing... then there is a country which...'. In both cases, a range of different (less than maximally natural) quantifier meanings are available for use, and a given linguistic community's immediate acceptance of claims like the above above helps determine which of these is relevant to their words. Thus no violation of the apparent parity between existence claims about mathematical objects and existence claims about ordinary objects is required.

introducing Ms give 'there are' a meaning on 'there are Ms' express a truth at all times in all possible worlds[14].

3.3. **Intelligibility worries.** Despite these advantages, the neo-Carnapian proposal sketched above approach faces an immediate worry arising from its appeal to Quantifier Variance$_M$. Are the kind of powerful variant quantifier senses which the neo-Carnapian explanation needs to posit really intelligible? In what sense would these variant notions be variant notions of existence?

Nearly everyone will allow that expressions like 'there is' can sometimes take on a *restricted* sense, as when someone says "all the beers are in the fridge". However many philosophers are inclined doubt the intelligibility of appeals to alternative quantifier-like senses for "∃" which are not mere restrictions of a fundamental most generous notion of existence which we use when doing ontology. For example, Wright and Hale claim not to understand "just what ...the postulated variant quantifier meanings [are] supposed to be."[2]. They say that philosophers who appeal to quantifier variance owe an explanation of, "why the allegedly different quantifiers which can all be expressed by the words 'there are' are quantifiers and ... how they differ in meaning."[2]. They grant that one can answer the first question by appealing to the existential quantifier's characteristic inferential role (as my formulation of Quantifier Variance$_M$ does). But they express grave doubts about whether the second question can be adequately answered[15].

[14]See [7].

[15]"The only obvious suggestion - that by introducing concepts of new kinds of objects (e.g. mereological sum, or number) we somehow enlarge the domain - is, in so far as it's clear, clearly hopeless. We cannot expand the range of our existing quantifiers by saying (or thinking) to ourselves: Henceforth, anything (any object) is to belong to the domain of our first-order quantifiers if it is an F (e.g. a mereological sum). For if Fs do not already lie within the range of the initial quantifier anything, no expansion can result, since the stipulation does not apply to them; while if they do, then again, no expansion can result, since they are already in the domain."[2]

4. Chalmers' proposal

In the rest of this paper I will attempt to address the worries about intelligibility of suitable variant quantifier senses mentioned above. In 'Ontological Anti-realism', David Chalmers provides an example of how to describe alternative quantifier senses which will be the point of departure for my own proposal. In this section, I will quickly summarize Chalmers' proposal, omitting certain features which are important to the larger project of 'Ontological Anti-realism' (using variant quantifier sense to provide supervaluationist truth conditions for statements in the metaphysics room) but irrelevant to the task currently at hand. I will then note a problem which arises if we try to use this story to explicate the kind of variant quantifier meanings which the neo-Carnapian explanation of mathematicians' freedom to introduce new objects mentioned above needs to posit.

One can think about Chalmers' account of what variant senses of the quantifier 'might be like' as combining two things: a (broadly) Fregean idea about what kind of explanation should suffice to defend the intelligibility of variant quantifier senses (which my own proposal will copy), and a specific set theoretic example of such an explanation (which my own proposal will replace).

The Fregean idea is this: one can satisfactorally explain the meaning of a logical connective merely by explaining how this connective systematically contributes to the truth or falsity of sentences in which it figures. This idea is quite plausible - for if one denies it, it becomes hard to see how we could give any explanation for the meaning of &, $\vee$ etc. If this idea is correct, we can answer Wright and Hale's challenge (merely) by giving a systematic story about how these variant quantifier senses contribute to the truth conditions for whole sentences. Thus one can explain variant quantifier senses merely by providing a "tool for helping to understand the conditions under which various sorts of existence assertions are true or correct" which (as Chalmers explicitly reminds us) "...should not be read as offering an account of the logical form of existence sentences, and need not be read as offering an account of the propositions expressed by these sentences, or as a conceptual analysis of these sentences."[5]

Chalmers implements this Fregean idea as follows. He describes variant quantifier senses $\exists_c$ by associating each such sense with a "furnishing function" F_c. This furnishing function F_c associates each possible world w with a set theoretic model $F_c(w)$ which specifies how many objects will count as 'existing' in the relevant sense $\exists_c$ at w, and how various properties will apply to these objects. Thus, for example, if it is true to say in shop class that there are holes, then the furnishing function associated with English as spoken in shop class will associate the actual world with a set theoretic model which assigns the property HOLE a non-empty extension. In this way, it assigns each possible world to a set theoretic model which provides a "catalog of the objects which are taken to exist in [that] world". Note that, crucially, the 'extension' which a furnishing function assigns to the property HOLE at a given world might not be a set of holes, but could instead be a set of integers.

Appealing to furnishing functions allows us to use our own *current* sense of the quantifier to systematically describe truth conditions for utterances employing alternative senses of the quantifier, as follows. The truth or falsehood of a proposition ϕ employing some alternative quantifier meaning $\exists_c$ at a possible world w can be determined by starting with the set theoretic model $f_c(w)$'s domain and extensions for atomic properties and relations, and then applying standard recursive rules for the logical connectives. Thus, for example, a proposition of the form $\exists_c F(x) \& G(x)$ will be true at exactly those possible worlds w such that $f_c(w)$ assigns some object to the extension of both properties F and G. Facts about the truth or falsity of sentences can then be built up from facts about truth at a world in the ordinary way.

I think Chalmers' story provides a clear and satisfying response to Wright and Hale's worries about what alternative quantifier-like meanings for "$\exists$" could be like. It allows us to use whatever sense our quantifier currently takes on to describe alternative quantifier senses which are not mere restrictions of this sense[16]. It also also allows us to explain how these senses are quantifier-like, by appealing to their inferential role as suggested above. For within any language that uses one of the variant senses proposed for "$\exists$" reasoning in accordance with standard first order logical inference rules will always be truth preserving. Thus all changes in quantifier meaning between the variants which Chalmers describes will preserve "$\exists$"'s usual inferential role.

Unfortunately, however, there are limits on the *range* of different quantifier senses which Chalmers' method allows one to describe. The problem is that his story specifies how each quantifier sense contributes to truth conditions for sentences by associating each possible world with set theoretic models (hence models whose domain is a set). This raises a serious problem for using it to describe quantifier senses on which one can truly assert the existence of structures which include proper class-sized pluralities of objects – like the hierarchy of sets itself[18]. This prevents us from using Chalmers'

[16]Note that merely imposing quantifier restrictions can never make a proposition of the form "$\exists x F(x) \& G(x)$"[17] go from being false to being true.

[18]Admittedly, by Skolem's theorem one can find a countable model for any (consistent) first-order theory in a countable language. However, intuitively it appears that we can mean the standard model of the numbers, which can't be specified in first order logic. Accordingly many philosophers have found it appealing to allow the use of more powerful logical vocabulary, such as second order logical quantification, which (unlike the vocabulary of first order logic) has the power to uniquely characterize the intended structure of the natural numbers . Accordingly, if we follow Chalmers' idea of taking logical vocabulary (including this powerful logical vocabulary) to apply in a straightforward way within the relevant domains, then appealing to Skolem's theorem won't guarantee the existence of suitable model. Furthermore, if you think that you can use this rich logical vocabulary

method to describe the full range of variant quantifier meanings which the neo-Carnapian explanation of mathematicians' freedom needs to posit.

5. Re-purposing Modal Structuralist Paraphrases

I will now propose a variant method for explaining what alternative quantifier senses "might be like" which is designed to avoid the problem noted above. This explanation copies Chalmers' strategy of describing variant quantifier meanings by explaining their contribution to truth conditions. However it eschews appeal to any particular definite totality of objects which all mathematical existence claims must be understood as having models within. Instead it draws on a powerful notion of 'logical possibility *given certain facts*' from the philosophy of mathematics literature[19] to provide intended truth conditions for sentences containing variant senses of the quantifier.

Nominalists like Geoffrey Hellman have appealed to a notion of logical possibility when attempting to provide nominalistic paraphrases for statements which appear to assert the existence of mathematical objects. Such paraphrases display what (the nominalist claims is) the true logical form of mathematicians' utterances – and thus show that these statements do not commit one to the existence of abstract objects. Like other nominalist accounts, Hellman's proposal faces a worry about denying the apparent uniformity of logical structure of mathematical and non-mathematical existence claims noted above.

However, we can avoid these problems if we use Hellman's paraphrases merely as a tool for explaining how variant quantifier senses contribute to

to categorically describe the hierarchy of sets itself, then it must be possible to formulate coherent claims (corresponding to acceptable mathematical stipulations) which don't admit a set model.

[19]See, for example, [8] and [13]. The details of my presentation are most influenced by Hellman.

the truth conditions for speakers utterances, rather than as articulating the true logical structure of mathematical statements. Rather than saying, "there are numbers" expresses a proposition whose true structure is, say, a conditional claim about what is logically necessary given certain facts (and hence facing the objections to failure of uniform semantics noted above) we will allow that this statement has the same logical form as "there are birds". We say that expressions like 'there is' have a uniform meaning when uttered as part of mathematical, sociological and ornithological sentences in a given context (outside the metaphysics room). However we say that acts of stipulative definition can change which meaning this is – and we use Hellman-style paraphrases as a tool for explaining what these variant possible senses which the quantifier can take on are like.

To see how Hellman's strategy works, first consider the limited task of providing nominalistic paraphrases for statements of pure mathematics (i.e. statements which appear to *only* quantify over mathematical objects. Many philosophers of mathematics have been inclined to acknowledge to a notion of broadly logical possibility which allows us to distinguish intuitively 'coherent' descriptions of mathematical structures for study (like second order Peano Arithmetic) from unacceptable ones (like naive set theory).

This notion of logical possibility resembles standard notions of having a set theoretic model, except for two important points. First, one can formulate $\Diamond\phi$ claims in cases where ϕ uses richer logical vocabulary than the standard first order logical connectives (e.g. the second order logical quantifier) to describe a pattern of relationships between objects. Second, facts about logical possibility are taken to be primitive modal facts which do not to require grounding in the existence of 'witnessing' objects like set theoretic models, or the possibility of re-interpreting relation symbols and restricting the domain of quantifiers in such a way as to make the relevant sentence true. This is important because it allows us to honor the intuitive idea that, for any describable structure of abstract objects, it would be logically possible for there to be a strictly larger structure e.g. one which added a layer of classes to the original structure[20]

Given this notion of logical possibility (and suitable descriptions of mathematical structures) Hellman notes that one can provide correct truth conditions for *pure* mathematical claims using a modal conditional $\Diamond D \& \Box(D \rightarrow \phi)$, where $\Diamond$ means 'it is logically possible that' and $\Box$ means 'it is logically necessary that' and D categorically describes the intended structure of the relevant mathematical objects. Appealing to the above notion of logical possibility and necessity allows the above paraphrases to provide intuitively correct truth conditions for claims about coherently described mathematical structures even if it turns out that the universe is too small to contain any collection of objects related in the way that D describes.

Unfortunately, the simple paraphrase strategy above fails to capture intended truth conditions for such mixed mathematical statements like, 'there are a prime number of rats'. For even if there are *actually* a prime number of rats, this does not suffice to ensure the logical necessity of a conditional like 'if there are numbers and functions then there is a bijection between the rats and the numbers below some prime number'.

To handle this difficulty, Geoffrey Hellman proposes a more general paraphrase strategy which appeals to a notion of what is logico-mathematically possible and necessary 'given the physical facts'. This notion may initially seem confusing or unclear if we try to understand it in terms of metaphysically or physically possible worlds which add extra objects (where are the

[20]Also, if one takes the hierarchy of sets to have a definite structure which we can uniquely describe using the kind of logical vocabulary which we use to describe other mathematical objects, it allows us to say that this description requires something logically possible (and descriptions of larger structures also require something logically possible).

numbers located, how are these worlds different from the actual ones). However, if one accepts the notion of logical possibility as 'possibility with regard to the most general combinatorial constraints on how any objects can be related by any relations' sketched above, I think there is an obvious extension of this idea to a notion of logical possibility 'given' certain facts which can be used to articulate Hellman's proposal. Consider statements like KITTENS.

> KITTENS: Given what kittens and blankets there are, it is logically impossible that each kitten slept on a different blanket last night.

I think there's a clear (and indeed natural) reading of this sentence on which will be true iff there are more kittens than blankets. If we accept a notion of unsubscripted logical possibility, where $\Diamond\phi$ facts are witnessed by the existence of a set theoretic model, facts about logico-mathematical possibility $\Diamond_{R1...Rn}$ given certain facts track are witnessed by the existence of a model *which preserves the actual extension of certain predicates* $R_1...R_n$.

Given use of some notion like this, one can capture intended truth conditions for (many) mixed statements like 'there are a prime number of rats' by a paraphrase of the form below, where (D is a categorical description of the structure of the numbers and) one talks about what is logically necessary given the facts about how some physical predicates apply.

$\Box_{rats,cats}(D \rightarrow \text{there are a prime number of rats})$[21]

6. An Example of this Method

Now let us turn to the details of what such a neo-Carnapian explanation of mathematicians' freedom might look like. In this section I will give a simple example of such an explanation.

[21]see below for a more detailed treatment of this example.

This story begins with the idea that we can think about mathematicians' choices to adopt theories positing new kinds of mathematical objects as behaving like acts of stipulative definition. I take such acts of stipulative definition to involve at least two elements. First there is a sentence S[22] whose truth one attempts to secure while giving meaning to whatever terms are being stipulatively (re)defined. In the case of stipulations introducing new kinds of mathematical objects, this sentence S would specify how the mathematical objects being introduced are to related to one another (and perhaps also how these objects are to relate to various previously understood mathematical objects).

Second, there is a choice of which vocabulary involved in the relevant sentence S is supposed to be (re)defined by one's act of stipulative definitions vs. which terms' meaning is supposed to be held fixed. Thus, for example, one might introduce a term like 'bachelor' by an act of stipulative definition which puts forward the sentence "$\forall x(\text{bachelor}(x) \leftrightarrow [\text{man}(x) \& \neg \text{married}(x)])$" together with permission to modify the meaning of the relation symbol 'bachelor()' but not 'married()', 'man()' or any of the logical connectives. On the controversial neo-carnapian approach to stipulative definitions which I will now outline, it will be possible for some successful stipulative definition to change the meaning of '$\exists$' as well as changing the meaning of various relation symbols.

To discuss these issues more formally, I will consider the effects of stipulations made by people speaking a logically regimented (interpreted) language that contains finitely many meaningful atomic relation symbols and no truth predicate. I will further suppose that this language contains a list of predicates $P_1(x), P_2(x) \ldots P_n$ which behave like an exhaustive list of kind terms,

[22] or an algorithmically listable collection of sentences or syntactic method of making inferences

in the following sense: '$\forall x(P_1(x) \vee P_2(x) \ldots P_n)$' expresses a metaphysically necessary truth in I_0. I have suggested that one can think of acts of stipulative definition as determining an ordered triple as follows:

- A set of sentence(s) S whose truth is being stipulated.
- A set of of atomic relation-symbols $R_1, ...R_m$ whose application this act of implicit definition is not permitted to modify
- A specification of a 0 or a 1 representing whether the stipulation is permitted to change the meaning of the quantifier.

I will say that making a stipulation $\langle S, R_1 \ldots R_n, 1\rangle$ while speaking I_0 will is **viable** iff $\Diamond_{R_1 \ldots R_n} S$ comes out true in I_0[at all possible worlds].

Now (my simple theory of mathematical stipulations says) making a viable stipulative definition $\langle S, R_1 \ldots R_n, 1\rangle$ while speaking I_0 would shift one to speaking an ideolect I_1 such that:

ϕ is true in I_1 if $\Box_{R_1 \ldots R_n}(S \rightarrow \phi)$ is true on I_0

To see how this proposal works more concretely, imagine that we start out speaking a base ideolect I_0 which does not 'talk in terms of' numbers[23]. Now suppose that we now want to discuss the possibility of starting to talk in terms of numbers.

One can give a categorical description of the intended structure of the numbers (how number(), S(), plus(), times() apply) using first order logical connectives and the subscriptable $\Diamond$ operator - I will call this $PA_\Diamond$[24]. One can then use this to formulate a description of the structure of the numbers which essentially says: there are numbers related to one another as per $PA_\Diamond$, all objects are are either numbers or of some type $P_1 \ldots P_n$, the numbers

[23]It may or may not talk in terms of other abstract mathematical objects like sets.

[24]In essence appeals to the combinatorially possible applications for a predicate play the role of appeals to second order logic

are distinct from all the different types of objects $P_1 \ldots P_n$ which we are currently talking in terms of, and S(), plus(,) times(,) only apply to numbers.

$$\text{NUMS: } PA_{\Diamond} \& \forall x[P_1(x) \vee P_2(x) \ldots P_n(x) \vee \text{number}(x)] \& \forall x[P_1(x) \vee P_2(x) \ldots P_n(x) \rightarrow \neg\text{number}(x) \& \neg \exists y S(x,y) \& \neg \exists y S(y,x) \& \ldots]$$

Accordingly, I will consider an act of stipulative definition which attempts to secure the truth of NUMS, and enjoys permission to implicitly (re)define expressions like 'number()', 'successor(,)', 'abstractObject()' and '∃' but not to change the meaning (in the sense of adding objects to the extension of) any of the finitely many remaining atomic relation symbols like 'dog()' or 'hole()' or 'is located at(,)'.

The simple theory of stipulation above says making this stipulative definition while speaking I_0 would shift one to speaking a related language I_1 with the same formal syntax as I_0 such that:

$$\phi \text{ is true in } I_1 \text{ if } \Box_{R_1 \ldots R_n}(NUMS \rightarrow \phi) \text{ is true on } I_0$$

To see how this account yields intuitively correct truth conditions for a broad range of sentences, let us consider some examples.

First consider a purely mathematical statement about the numbers like PRIMES: 'There are infinitely many prime numbers[25]' Because $NUMS$ includes a categorical description of the numbers $PA_{\Diamond}$, it's logically necessary that if number(), S(), plus(,) etc apply as per $NUMS$ then there are infinitely many prime numbers. Thus we have $\Box(NUMS \rightarrow PRIMES)$. We also have the weaker statement that it is logically necessary, given the application of the relations $R_1 ... R_n$, that $NUMS \rightarrow PRIMES$. Thus we have $\Box_{R_1 \ldots R_n}(NUMS \rightarrow PRIMES)$ as desired.

[25]Strictly speaking this statement and 'There are a prime number of rats' are imprecise natural language descriptions requiring translation into $\mathscr{L}$.

More generally, the modal structuralist strategy captures the important (and historically often hard to handle!) intuition that facts about mathematical objects can, in some cases, outrun our ability to prove claims about these objects. Intuitively we'd like all statements in the language of number theory to have definite truth conditions. $NUMS$ includes a categorical description of the intended structure of the numbers, so for every sentence ϕ in the language of number theory, either $\Box(NUMS \rightarrow \phi)$ or $\Box(NUMS \rightarrow \neg\phi)$. Accordingly the paraphrase indicated above does indeed ensure that for every sentence ϕ in the language of number theory either ϕ or $\neg\phi$ comes out true.

Next consider a purely physical sentence like "$\exists x$ rat(x)". Suppose that there are exactly 11 rats. My simple story correctly predicts that "$\exists x$ rat(x)" will express a truth in I_1, as follows. There are eleven rats. So, given what rats there are, it's logically necessary that $\exists x rat(x)$. Accordingly it's logically necessary, given the facts about how 'rat()' and *a range of other relation symbols* in $R_1...R_n$ apply, that there $\exists x rat(x)$. Thus $\Box_{R_1...R_n} \exists x$ rat(x) and $\Box_{R_1..rats()..R_n}[NUMS \rightarrow \exists x \text{ rat}(x)]$.

Finally consider the simple applied mathematical statement, 'There are a prime number of rats' (i.e., 'It would be logically possible, fixing what rats and numbers there are, for the rats to be bijectively paired with an initial segment of the natural numbers up to some number $n-1$ where n is prime). This statement comes out true because, in essence, the existence of 11 rats makes it broadly logically necessary, given what rats there are, that if there are numbers as per NUMS as well, then it is logically possible (given what rats and numbers there are) to pair up rats with the numbers below 11, or without 11 being prime[26].

[26]By 11 I mean, of course, the 11th successor of the number (0) which is not a successor of anything.

Thus I think the story above yields intuitively appealing verdicts about how acts of ontologically empowered stipulation introducing new kinds of mathematical objects can change the meaning of our words in various contexts[27].

Like Chalmers' set theoretic proposal, the above story explains the meaning of alternative senses for the quantifier by describing how these senses systematically contribute to truth conditions for a larger unit (in this case, a sentence). It also lets us explain why these alternative notions are quantifier-like, by noting that standard inference rules for the quantifiers will remain truth-preserving[28].

Unlike Chalmers' account however, the above story lets us describe senses of 'exists' which demand the 'existence' of mathematical structures which have no model within the mathematical structures we accept. Appealing to the fundamentally modal notion of combinatorial possibility allows us to crisply describe truth conditions for statements involving variant ontologically profligate senses of the quantifier. In particular, it allows us to explain how mathematical stipulations characterizing structures which are too large

[27]Hartry Field has pointed out Hellman faces a problem about how to capture intended truth conditions for more complex statements of applied mathematics like 'this backpack weighs 3.7 times more than that one', without appealing to infinitely many atomic predicates. This problem does not apply to us insofar as we are using one language which talks in terms of mathematical objects to capture truth conditions for statements in a language which talks in terms of strictly more mathematical objects.
To see how this story avoids Field's challenge to Hellman's original theory, note that the technique above lets one capture stipulations which attempt hold fixed the meaning of current mathematical vocabulary like 'realNumber()' and 'hasMassRatio(,,)'. Thus if we are currently speaking a language which talks in terms of numbers, and uses relationship to mathematical objects to measure ratios, this poses no problem for using my modal structuralist strategy to explain how making stipulations introducing new abstract objects would change the meaning of our quantifiers.

[28]Note that these rules remain truth preserving even in cases where my theory does not generate classical truth conditions, but rather leaves truth-value gaps because the relevant stipulation fails to characterize a unique structure.

to have any (standard) models within the hierarchy of sets could nonetheless succeed in changing the meaning of our quantifiers in such a way as to ensure their own truth.

In this way I think appeals to a notion of logical possibility given certain facts help us articulate an attractive neo-Carnapian explanation of mathematicians' freedom to introduce new objects: one which combines the smooth semantics of set theoretic foundationalism with nominalists' avoidance of arbitrary limits on mathematicians' freedom.

7. Conclusion

In this paper I have attempted to develop a Neo-Carnapian story which uses the idea that the quantifier can take on different senses in different contexts to explain mathematicians' freedom to stipulate. A major source of objection to this approach is that no sense could be made of the kind of alternative, extremely ontologically profligate, senses for the quantifier which this account required one to posit. I suggested that Chalmers' (essentially) set theoretic description of variant quantifiers' contribution to the truth conditions for sentences sufficed to address worries about the intelligibility of ontologically profligate senses for the quantifier, but did not allow us to make sense of of the extremely ontologically profligate quantifier senses which this neo-carnapian explanation needs.

I then outlined a variant method for describing alternative ontologically profligate senses of the quantifier which replaced appeals to set theory with appeals to a notion of broadly logical possibility. This alternative sense of the quantifier allowed one to describe the behavior of a sufficiently broad range of alternative quantifier senses to vindicate the original idea that all coherent extensions of a given mathematical practice would express truths. Adopting this strategy lets us answer Wright and Hale's worry by clearly and non-paradoxically describe the kind of very ontologically profligate alternative quantifier senses which the quantifier variance explanation needs to posit.

References

[1] Paul Benacerraf. Mathematical truth. *Journal of Philosophy*, 70:661–80, 1973.

[2] Crispin Wright Bob Hale. The metaontology of abstraction. In Ryan Wasserman David Chalmers, David Manley, editor, *Metametaphysics: New Essays on the Foundations of Ontology*. Oxford University Press, 2009.

[3] George Boolos. *Logic, logic and logic*. Harvard University Press, 1999.

[4] John Burgess. Review of stewart shapiro, philosophy of mathematics: Structure and ontology. *Notre Dame Journal of Formal Logic*, 40(2):283–91.

[5] David Chalmers. Ontological anti-realism.

[6] Julien Cole. Towards an institutional account of the objectivity, necessity, and atemporality of mathematics. *Philosophia Mathematica*, 2013.

[7] Iris Einheuser. Counterconventional conditionals. *Philosophical Studies*, 2006.

[8] Geoffrey Hellman. *Mathematics Without Numbers*. Oxford University Press, USA, 1994.

[9] Eli Hirsch. *Quantifier Variance and Realism: Essays in Metaontology*. Oxford University Press, 2010.

[10] Oystein Linnebo. Platonism in the philosophy of mathematics. Stanford Encyclopedia of Philosophy, 2011.

[11] Hillary Putnam. Mathematics without foundations. In Paul Benacerraf and Hilary Putnam, editors, *Philosophy of Mathematics, Selected Readings*. Cambridge University Press, 1983.

[12] Stewart Shapiro. Classical logic, Winter 2013.

[13] Stuart Shapiro. *Philosophy of Mathematics: Structure and Ontology*. Oxford University Press, USA, 1997.

[14] Theodore Sider. Ontological anti-realism. In *Metametaphysics: New Essays on the Foundations of Ontology*. Oxford University Press, 2009.

[15] Gabriel Uzquiano. The price of universality. *Phil Studies*, 1996.