Which Modal Machinery Should the Set Theoretic Potentialist Use?

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Abstract

Accepting some form of potentialist set theory promises to help us solve puzzles about the intended height of the hierarchy of sets. However, philosophers have developed two different schools of potentialist set theory: minimalist and dependence based formulations of potentialist set theory. In this paper, I will argue that minimalist formulations of potentialism have some important advantages over dependence based formulations.

1 Introduction

Accepting some form of potentialist set theory promises to help us solve puzzles about the intended height of the hierarchy of sets. However, philosophers have developed two different schools of potentialist set theory, which I will call (following Neil Barton[1]) minimalist[18, 9, 10, 2] and dependence theoretic[15, 17, 14, 12, 21] formulations of potentialist set theory.

In this paper I will argue that minimalist potentialism best explicates mathematicians' set theoretic talk. In §2 I will review the motivations for potentialist set theory and contrast existing developments of minimalist and dependence theoretic versions of potentialism. In §3 I will argue that minimalist potentialism has two key advantages over dependence theoretic approaches. First, the dependence theorist faces an immediate puzzle about answering 'how many pure sets there actually?' in a principled fashion. Going minimalist lets us avoid this problem. Second, going minimalist has an advantage of conceptual economy: we can use the familiar and (I will argue) independently needed notion of logical possibility rather than introducing a novel and arguably somewhat unnatural/underspecified notion of interpretational possibility. Finally in §4 I address some possible worries about the minimalist approach.

2 Background

2.1 Motivations for Potentialist Set Theory

So, let's begin by reviewing what potentialism about set theory is, and how this view is commonly motivated. Recall that, after the discovery of Russell's paradox, set theorists embraced an iterative hierarchy conception of sets. On this view, all sets exist within a hierarchy of different layers (that satisfy the well ordering axioms). There's the empty set at the bottom. And each layer of sets contains sets corresponding to 'all ways of choosing' from sets generated below that layer (i.e., all subsets of the set of sets occurring at prior layers).

This conception of how the sets are supposed to be structured precisely characterises the intended width of the hierarchy of sets. But what about the height of the hierarchy of sets? How many layers of sets are there? Here a puzzle arises that motivates potentialist set theory. Naively, it is tempting to say that the hierarchy of sets is supposed to extend 'all the way up' in a way that guarantees it satisfies the following principle

> **Naive Height Principle:** If some objects are well-ordered by some relation $<_R$, there is an initial segment of the hierarchy of sets whose structure mirrors that of these objects under relation $<_R$ (in the sense that the objects related by $<_R$ could be 1-1 order preservingly paired onto the layers in this initial segment).

But this assumption leads to contradiction via what's called the Burali-Forti paradox[1]. So, in contrast to the fact that we seem to have a precise and logically coherent conception of the intended *width* of the hierarchy of sets, we don't seem to have any analogous conception of its intended height (that remains once the naive and paradoxical idea above is rejected). And it seems arbitrary to say that the hierarchy of sets just happens to stop somewhere: that it has a certain height which doesn't follow from anything in our conception of the structure of the hierarchy of sets.[2]

Potentialist approaches to set theory provide a popular response to the above problem. In a nutshell, potentialists reject the idea that there is a unique intended point at which the hierarchy of sets stops. Potentialists reinterpret ordinary set theoretic statements, so as to replace apparent quantification over a single intended hierarchy of sets with claims about how it would be (in some sense) possible for intended-width initial segments of the hierarchy of sets to be extended.

However, there are a number of different ways of implementing the above potentialist idea. I will now flesh out the skeletal description of potentialism given above by describing both minimalist and dependence theoretic approaches to potentialism in some detail.

The main difference between the two styles of potentialist set theory I'll be comparing is this. Dependence potentialism interprets set theory as talking

[1] If we consider the relation $x <_R y$ 'iff x and y are both layers in the hierarchy of sets and x is below y or y is the Eiffel tower and x is a layer' we see that the above naive conception of the hierarchy of sets cannot be satisfied. We have a sequence of objects that is strictly longer than the hierarchy of sets, contradicting the naive conception of sets. We know the sequence of objects related by $<_R$ is strictly longer than the layers of the hierarchy of sets because it's a theorem of ZFC that no well ordering is isomorphic to a proper initial segment of itself.

[2] Note that the problem here is not simply that it might be impossible to define the intended height of the hierarchy of sets in other terms. After all, every theory will have to take some notions as primitive.

Instead, we find ourselves in the following situation. Our naive conception of absolute infinity (the height of the actualist hierarchy of sets) turns out to be incoherent, not just analyzable. And, once we reject this naive conception, there's no obvious fallback conception that *even appears* to specify a unique height for the hierarchy of sets in a logically coherent way.

about what *sets* there (in some sense) could be. In contrast, the minimalist approach to set theory (which I favor) interprets set theory as talking about how there could be objects (of any kind) satisfying certain set theoretic axioms (and, thus, having the intended structure of an initial segment of the hierarchy of sets).

We will see that cashing set theory out in terms of what sets there could be turns out to allow the dependence theorist to produce simpler logical regimentations of set theoretic sentences. But it does this at the cost of requiring introduction of a largely novel modal notion (e.g., Linnebo and Studd's 'interpretational possibility') expressing a sense of possibility on which it's contingent how many pure sets exist, but it's necessary that the sets that do exist satisfy certain axioms that aren't logical truths. In contrast, the minimalist potentialist can formulate their paraphrases using a notion of logical possibility that we have independent reason to accept.

2.2 Minimalist Potentialism

The minimalist approach to potentialist set theory (which I will be advocating) traces back to work by Hilary Putnam. In [19] Putnam sketches a way of thinking about set theory in terms of modal logic: as talk about what 'models' of set theory are, in some sense, *possible* and how such models can be extended.

He introduces a notion of being a standard model of set theory, which is a model of set theory closed under subsets, i.e., a hierarchy of sets having full width and no infinite descending chains under $\in$[3]. Putnam says that we can 'make this notion concrete' by thinking of models as physical graphs consisting of

[3]Specifically, Putnam writes "[A concrete] model will be called standard if (1) there are no infinite-descending 'arrow' paths; and (2) it is not possible to extend the model by adding more "sets" without adding to the number of "ranks" in the model. (A 'rank' consists of all the sets of a given-possibly transfinite-type. 'Ranks' are cumulative types; i.e., every set of a given rank is also a set of every higher rank. It is a theorem of set theory that every set belongs to some rank.)"

pencil points (or the analog of pencil points in space of some higher cardinality) and arrows connecting these pencil points. And he "ask[s] the reader to accept it on faith" that we can express the claim that some model is standard in this way "using no 'non-nominalistic' notions except the '$\Box$"' (where $\Box$ denotes the logical necessity operator).

With this notion of a concrete model in place, Putnam suggests that we can understand set-theoretic statements as claims about what such models are possible, and how they can be expanded. For example, he proposes that we can paraphrase a set-theoretic statement of the form '$(\forall x)(\exists y)(\forall z)\phi(x, y, z)$' where ϕ is quantifier free, as saying that, if G is a standard concrete model, and p is a point within G, then it is possible that there is a model G' which extends G, and a point y within G' such that necessarily, for any model G'' which extends G' and contains a point z, $\phi(x, y, z)$ holds within the concrete model G''. And we can treat arbitrary quantified statements in set theory in an analogous fashion.

In [9] and later work Geoffrey Hellman develops Putnam's picture by suggesting that we should understand the key modal notion $\Diamond$ in Putnam's potentialist set theory to express a primitive modal notion of logical possibility, which I will discuss in more detail below. In [9] Hellman notes that we can cash out Putnam's appeal to 'standard models' of set theory by saying that standard models are models which satisfy ZFC_2 (i.e., the version of standard ZFC set theory which replaces the inference schemas of Replacement and comprehension with corresponding second-order axioms)[4][5].

Thus, for example, a minimalist potentialist might paraphrase "$(\forall x)(\exists y)(x \in$

[4]So, for example, ZFC expresses comprehension via an axiom schema which contains an axiom for every formula ϕ in the language of set theory. In contrast, by using second-order logic one can state a single comprehension axiom as follows $(\forall x)(\forall C)(\exists y)(\forall z)(z \in y \leftrightarrow z \in x \wedge C(z))$. The same goes for the first-order axiom schema of Replacement and its second-order analog.

[5]In later work like[10], Hellman notes that we can do similar work using plural quantification and mereology. And Berry[2] argues that one can do it using a motivated generalization of the logical possibility operator itself.

$y)$" , as follows (where quantification over all V_i as shorthand for quantification over all second order objects[6] X, f satisfying some axioms like ZFC_2 (in the sense that $ZFC_2[set/X, \in /f]$) which ensure that the objects satisfying X have the intended structure of a hierarchy of sets when considered under the relation f.

$\Box(\forall V_1)(\forall x)[x \in V_1 \rightarrow \Diamond(\exists V_2)(\exists y)(y \in V_2 \wedge V_2 \geq V_1, \wedge x \in y)]$

Note that if we were to fully expand out the notation above, the resulting sentence would only use modal and logical primitives (not including either set or $\in$).

Adopting some such potentialist approach to set theory can help us dispel the Burali-forti worries about the intended height of the hierarchy of sets discussed above. For, the potentialist can understand set-theoretic talk without imposing or positing arbitrary limits on the size of structures (as we would do if we just stipulated a point at which the hierarchy of sets stopped, or inferring that it must stop somewhere) in a way that seems faithful to our intuitions about the generality of set-theoretic reasoning[7].

Later work on minimalist potentialism by Hellman[9] and Berry[2] develops Putnam's idea by appeal to a notion of logical possibility (which has been argued to be an independently attractive primitive). Hellman originally uses logical possibility, together with second order quantification (which later gets replaced by plural quantification and mereology used to simulate second order relation quantification). Berry uses a generalization of the logical possibility operator.

2.3 Dependence Potentialism

A different, dependence-theoretic, approach to potentialist set theory is currently somewhat more popular than minimalist potentialism (which interprets

[6](or pluralities simulating them)

[7]In particular, (before thinking about the paradoxes) we'd hoped for set theory to be general in the sense that every possible structure will have a copy somewhere in the sets.

set theory as talking about what any system of objects satisfying certain axioms would be like). Dependence theoretic potentialists acknowledge the existence of special objects called 'sets' (like traditional actualists), but interpret set theory as talking about what sets could (in some sense) be formed. Inspiration for this style of potentialism comes from [15, 16, 17]. However, I will focus on the more detailed recent developments of this approach by Linnebo[11, 12, 14] and Studd[21]. Linnebo and Studd allow that whatever sets (if any) exist, are metaphysically necessary objects. However they develop a potentialist set theory which makes claims about how it would be 'interpretationally' possible for a hierarchy of sets to grow. Claims about the interpretational possibility of such growth involve something like successively reconceptualizing the world so as to think and/or speak in terms of more and more sets (taller and taller actualist hierarchies of sets).

In [13] Linnebo explains the contrast between his preferred Dependence approach to potentialist set theory (what he calls Parsonian potentialism) and the Minimalist potentialism discussed above (which he calls Putnamian potentialism) as follows.

> [On a Dependence Potentialist approach to set theory] the idea is not to 'trade in' one's mathematical objects in favor of modal claims about possible realizations of structures but rather to locate some modally characterized features in the mathematical objects themselves. The mathematical universe is not 'flat'. Rather, some of its objects stand in relations of ontological dependence, and the existence of some of its objects is merely potential relative to that of others.
>
> 'A multiplicity of objects that exist together can constitute a set, but it is not necessary that they do. Given the elements of a set, it

is not necessary that the set exists together with them. ... However, the converse does hold and is expressed by the principle that the existence of a set implies that of all its elements.' (Parsons, 1977, pp. 293–4)[15]

So, the Dependence Potentialist takes the term 'set' to have pre-existing meaning (and facts about the essential nature of sets to do critical work in their theory), while (as we have seen) the term 'set' is completely eliminable from the Minimalist's theory. In this way, they resemble the advocate of mainstream actualist approaches to set theory. However, unlike the mainstream actualist approach to set theory, the dependence theorist holds that it is (in some important sense) contingent how many sets exist.

Dependence Potentialist paraphrases of set-theoretic sentences have a similar large-scale structure to Minimalist paraphrases, replacing $\exists$ claims with $\Diamond$ claims and $\forall$ claims with $\Box$ claims. However, they take the relevant notion of possibility to concern what sets could (in some relevant sense) be formed. And (as we will see) Dependence Potentialists don't write any description of the iterative hierarchy structure into their potentialist paraphrases; instead they take the fact that whatever sets exist form (something like) iterative hierarchy to fall out of — and be explained by — the nature of sethood.

Rather than talking about whether there objects satisfying certain axioms describing the intended structure of the hierarchy of sets, they just talk about what sets could exist. They are able to do this because they take the fact that whatever sets exist form an iterative hierarchy structure (and hence satisfy these axioms) to flow from facts about the essences of sets.

- e.g., Sets have their elements necessarily (so a set can't be formed before its elements have been formed), and sets are extensional (i.e., two sets are identical iff they have the same elements).

The Dependence Potentialist imagines a hierarchy of sets which could grow (with new sets somehow being formed) as follows. The empty plurality always exists. So an empty set could be formed. Form it. Now there's a plurality xx whose sole member is the empty set, so a set $\{\{\ \}\}$ could be formed. Form that. Now that both these sets exist, there are four pluralities xx of sets. And two of them correspond to sets we don't already have. So we could form $\{\{\{\ \}\}\}$ and $\{\{\{\ \}\}, \{\}\}$ etc.

This turns out to let them give simpler paraphrases for set theoretic claims. For example, recall how we said that a minimalist potentialist might paraphrase "$(\forall x)(\exists y)(x \in y)$" , as follows (where quantification over all V_i as shorthand for quantification over all second order objects[8] X, f satisfying some axioms like ZFC_2 (in the sense that $ZFC_2[set/X, \in /f]$) which ensure that the objects satisfying X have the intended structure of a hierarchy of sets when considered under the relation f.

$\Box(\forall V_1)(\forall x)[x \in V_1 \rightarrow \Diamond(\exists V_2)(\exists y)(y \in V_2 \wedge V_2 \geq V_1, \wedge x \in y)]$

And if we were to fully expand out the notation above, the resulting sentence would only use modal and logical primitives (not including either set or $\in$). In contrast, the Dependence theorist would formulate same claim more simply as saying something like the following (using 'set' as a meaningful primitive).

$\Box(\forall x)[set(x) \rightarrow \Diamond(\exists y)(set(y) \wedge x \in y)]$

And the Dependence potentialist thinks there are two readings of set-theoretic talk. In philosophical contexts like the paragraph above, we can quantify over the sets that literally exist. However, in mathematical contexts, talk which appears to say that certain sets exist is always shorthand for corresponding claims about what sets could be formed.

But what modal notion can the dependence theorist use? For example, it is presumably not *metaphysically* possible for more pure sets to exist? Recent

[8](or pluralities simulating them)

dependence theoretic potentialists have appealed to a notion of interpretational possibility suggested by [6]. Linnebo philosophically develops this notion in[14], and Studd[21] references Linnebo and invokes a generalization of Linnebo's notion interpretational possibility.

Very crudely, the idea is to say that it's interpretationally possible for more sets to be formed in the sense that we could change our language and/or modes of thought to carve the world up into more sets by adopting a sequence of Fregean abstraction principles. These Fregean abstraction principles relate the newly introduced objects to antecedently understood objects and pin down individuation criteria for these new objects. For example, in Frege's classic case, if you are already talking about lines, you can start talking in terms of the abstract objects we call 'directions', by adopting the abstraction principle that two lines have the same direction iff they are parallel. Accordingly it is interpretationally possible for there to be directions. Similarly, the dependence theorist takes it to be interpretationally possible for there to be more sets than there actually are, because they take it that we could start thinking in terms of more sets than we currently are by adopting a Fregean abstraction principle relating these new sets to the sets we are currently thinking in terms of.

As Linnebo puts it, in his dependence theoretic potentialist paraphrases the "modal operators $\Box$ and $\Diamond$ ... describe how the interpretation of the language can be shifted — and the domain expanded — as a result of abstraction."[14]. In particular, $\Diamond\phi$ is true iff you could make ϕ true via some well-ordered sequence of acts of reconceptualizing the world via adopting abstraction principles (whether or not it would be metaphysically possible for anyone to make such a sequence of abstractions).

Note that the adoption of such abstraction principles doesn't bring anything into being — whether it be a physical object or an abstract object. Rather it involves "reconceptualizing" the world. Also note that Linnebo's notion of interpretational possibility only allows reconceptualizations which recognize more objects, not ones which remove objects we currently recognize. Since his notion of possibility only allows the world to grow, it doesn't satisfy S5 (unlike logical possibility)[9]. Linnebo accepts the converse Barcan Marcus formula as true with regard to interpretational possibility.

3 Advantages of Minimalist Potentialism

With this contrast between minimalist and dependence based potentialist set theory in mind, I will now argue that the minimalist approach has two important advantages.

3.1 How many sets are there actually?

First, as we've just noted, Linnebo and Studd invoke a notion of interpretational possibility to cash out claims that more pure sets 'could be formed'. To say that certain sets 'could be formed' expresses an idea about how we could (in principle) change our language to talk in terms of more sets: how the interpretation of our language could be changed.

However, there's still some awkwardness about how many sets (if any) we are currently thinking in terms of. I will argue that the answer Stuff proposes to this question is unsatisfactory, and it's unclear whether a principled answer can be given. This reveals a sense in which the notion of 'could think in terms of' used by Linnebo and Studd to explain interpretational possibility is, at best, under-specified and perhaps impossible to sharpen in a way that's motivated and non-arbitrary. One might even argue that the dependence theorist trades in the

[9]Speaking in terms of Kripke models, when it comes to interpretational possibility only worlds that preserve or add to the objects existing in a world w_0 are accessible from w_0.

original arbitrariness problem motivating potentialist set theory (arbitrariness in answering 'how tall is unique intended hierarchy of sets?') for continuum many versions of that problem (how tall is the hierarchy of sets I'm thinking in terms of today? Tomorrow? The day after that?).

So let's review and consider the theory Studd proposes to answer to the questions 'how many sets are there actually?' (i.e., how many sets is my current language is talking in terms of?). Studd proposes a theory about how mathematicians can switch from thinking in terms of fewer sets to thinking in terms of more sets, in the course of ordinary mathematical practice

In Chapter 8 of [21] Studd sketches a story about how people engaging in ordinary mathematical practice could unknowingly change their quantifier meanings and come to talk in terms of a progressively larger actualist hierarchy of sets as follows. First considers a situation where people *knowingly* start talking and thinking in terms of extra sets. Imagine that some people who start out speaking a language Q that 'talks in terms of' a certain hierarchy of pure sets. And imagine that some of these people decide to split off from the main body of Q-language speakers and develop a new language E which talks in terms of extra sets.

Studd argues that this splinter group could achieve their ends by adopting certain principles, most importantly the inference schemas below for reasoning from claims in the old language Q (indicated below by putting 'Q:' in front of them) to claims in the new language E (indicated below by putting 'E:' in front of them), and vice versa. .

$$Q : things(vv) \Rightarrow E : thing(\{vv\})$$

$$Q : things(vv), Q : v \prec vv \Rightarrow E : v \in \{vv\}$$

$$Q : things(vv), E : v \in \{vv\} \Rightarrow Q : v \prec vv$$

Intuitively, these schemas embody the idea that each plurality vv of objects quantified over in the old language Q is supposed to form a set in the new language[10]. By accepting such inferences, our splinter group forces a charitable interpreter to interpret the quantifiers in their new language E as ranging over strictly more objects than they did in there original Q. By consciously adopting such inference rules relating a new ontologically inflationary language to the old one they start talking in terms of new objects.

Next Studd argues that unknowingly accepting inconsistent axioms of set theory (including the ones below)[11] can give rise to similar kind of expansionary quantifier meaning change:.

$$things(vv) \Rightarrow thing(\{vv\})$$

$$things(vv), v \prec vv \Rightarrow v \in \{vv\}$$

$$things(vv), v \in \{vv\} \Rightarrow v \prec vv$$

The above inference principles are inconsistent in a familiar Russellian way. They let you infer that, for any plurality of things vv, there's a set $\{vv\}$ whose elements are exactly the objects v in this plurality vv (written $v \prec vv$). But accepting the existence of this set (together with normal plural comprehension principles saying that, for any ϕ, there's a plurality vv of the objects such that ϕv) lets you derive the existence of the Russell set and hence contradiction.

[10]See page 235.
[11]See pg 239.

Studd argues that speakers endorsing the principles above would unwitting undergo quantifier meaning change, for the following reason. In general, a charitable interpreter can try to accommodate a speaker's reasoning by changing the domain of objects they take the speaker to quantify over[12] and the language they take them to be speaking. And it is more charitable to interpret speakers to be undergoing language change analogous to the switch from Q to E envisaged above, rather than saying something inconsistent. So, if meaning reflects charitable interpretation, accepting inconsistent principles along the lines above can produce a kind of unwitting quantifier meaning expansion in this way.

This is, I take it, Studd's proposal for how it could be true that (unbeknownst to us) our current quantifiers range over some steadily growing range of sets. He puts it forwards as the "basis for an idealized account of universe expansion applicable to the ordinary English speaker".

I don't think the above suggests a credible account of how we could come to talk in terms of one actualist hierarchy of sets rather than another (it doesn't seem to answer our question 'in virtue of *what* the actual hierarchy of sets have one height rather than another, according to interpretational possibility theorist?'). For one thing, surely we who live after the discovery of Russell's paradox, don't actually have the disposition to infer that arbitrary pluralities form a set. So the above story doesn't seem like a proposal for how contemporary set theorists (or ordinary English speakers deferring them) could come to talk in terms of more sets. Thus it's (at best) unclear to me that Studd's story describes people using our actual contemporary concept of set (and hence coming to talk in terms of more *sets*), as opposed to merely using the word "set" to express a different concept[13].

[12]Studd gives this example, "I utter '52% of people voted for Brexit' and we immediately limit the domain to exclude those who didn't turn out or were ineligible to vote"

[13]You and I might be suspicious about whether there's a principled distinction here. But note that the dependence theorist cannot. They need to say that it's interpretationally possible for there to be more sets than there are, but not for there to be sets that are elements of

Second, Studd's story leaves an obvious question about when and how quickly speakers are supposed to go through language change events he proposes (and thus fails to even suggest any principled answer to the question 'how many sets am I currently talking in terms of?'). If I lie around, having the inconsistent inference dispositions Studd mentions and not thinking about set theory for an hour, how many times should the charitable interpreter take my language to have changed during that time (every 5 minutes? every 10 minutes?)? Insofar as standing dispositions to make certain inferences (or to regard failure to make accept them when suggested as irrational) drive the above charitable interpretation, it is hard to see how one could give any non-arbitrary answer to the above question.

Thus, I think the key idea that we 'could think in terms of' more sets at the heart of Linnebo and Studd's interpretational possibility based dependence theoretic potentialism seems problematic because it's hard to imagined a principled attractive answer to 'how many sets are we thinking in terms of now?'. As noted above, one might even argue that the dependence theoretic potentialist trades in the original arbitrariness problem used to motivate potentialist set theory above (arbitrariness in answering 'how tall is unique intended hierarchy of sets?') for continuum many versions of that problem (how tall is the hierarchy of sets I'm thinking in terms of today? Tomorrow? The day after that?).[14]

themselves or the like. So they need to posit a kind of enduring core meaning to set which is preserved by language changes which make certain sentences about 'the height of the hierarchy of sets' change truth value, but not by language changes that make 'some set is an element of itself come out true.

[14]Perhaps the best strategy of the interpretational possibility theorist would be to say that we aren't currently thinking in terms of any sets. Rather mathematicians are engaged in a practice that's best understood potentialistically as making claims about how it would be possible for people engaged in a *(somehow) more explicitly actualist* practice could change their language. However neither Linnebo nor Studd seems to take this route. Perhaps this is because of the emphasis the interpretational possibility theorist puts on considering ways we could think in terms of/have languages that whose quantifiers range over *more sets*, rather than just using the word "set" a different way with a different meaning.

3.2 An Independently Motivated Modal Notion?

A second advantage I claim for minimalist potentialism concerns conceptual economy. Adopting minimalist potentialism has the advantage of letting us explicate set theory using a notion (of logical possibility interdefinable with entailment) which we have strong *independent* reason to accept as a modal primitive[15]. At first glance, one might argue that claims about logical possibility are merely shorthand for claims about the existence of set-theoretic models — so that it would be regress generating to use this notion to analyze set theoretic claims. However there are actually strong independent reasons pointed out in [7, 8, 3] (see also [5] 2.3 and Etchemendy [4]) for not doing this.[16]

A further benefit of accepting primitive logical possibility operator $\Diamond$ (as logical vocabulary) is that it lets us capture Boolos' intuition that there's something odd about identifying claims about logical possibility and validity with

[15]In contrast, to my knowledge, the accepting a primitive modal notion of interpretational possibility lacks strong independent motivation. Admittedly, Linnebo and Studd both use interpretational possibility to formulate theories about absolutely general quantification that make claims of general philosophy of language and metaontology. So it's not like this notion only gets used in formulating set theory. However arguably the main and perhaps only motivation for needing to deny the possibility absolutely general quantification (almost the only motivation presented in Studd's [21] and Fine's [6]) comes from the case of set theory itself. More general neo-Carnapian ideas about language change — that we could start to talk in terms of more or fewer objects (and perhaps sharpen the current meanings of our terms to do so) — have independent appeal and motivation. But I don't see a strong case that we need a separate interpretational possibility operator (distinguished from the logical possibility operator) and take all the objects and properties to have interpretational essences to express these more general neo-Carnapian impulses. Perhaps Linnebo's discussion of objections to ultra-thin objects is relevant here.

[16]Many philosophers have argued, as follows, that we shouldn't identify claims about logical possibility with claims about set-theoretic models.

The claim that what's actual is logically possible is central to the above notion of logical possibility (interderivable with validity), if anything is. For an argument to be valid surely at least requires that it doesn't *actually* lead from truth to falsehood.

However, if we think about logical possibility in terms of set-theoretic models, then the actual world is strictly larger than the domain of any set-theoretic model (e.g., because it contains all the sets). So it's not prima facie clear why we should infer from the fact that ϕ isn't be satisfied in any set-theoretic model, that ϕ isn't actually true. Thus we seem to antecedently grip a notion of logical possibility (interdefinable with validity) on which it's an open question whether every logically possible state of affairs has a set-theoretic model.

Now it is *currently* possible for mathematicians talking about *first order logical sentences* to replace talk of logical possibility with talk of set-theoretic models via the completeness theorem for first order logic[17]. However, as Boolos puts it, "it is rather strange that appeal must apparently be made to one or another non-trivial result in order to establish what ought to be obvious: viz., that a sentence is true if it is valid"[3].

set-theoretic claims (claims about the existence of set-theoretic models). We can agree with Boolos that, "one really should not lose the sense that it is somewhat peculiar that if G is a logical truth, then the statement that G is a logical truth does not count as a logical truth, but only as a set-theoretical truth.", and so reject cashing out claims about failures of logical truth/validity in terms of logically contingent claims about the existence of certain objects (even mathematical objects). If we treat the $\Diamond$ of logical possibility a primitive modal operator, and furthermore *logical* operator[18] we can secure the verdict that facts about logical possibility are themselves logically necessary truths.

Using logical possibility to formulate set theory (and then agreeing with Boolos that facts about logical possibility should turn out to be logical facts) also lets one affirm longstanding broadly logicist intuitions that there's a special connection between math and logic.[19]

Thus, I think there's appeal to accepting logical possibility a modal primitive and cashing out potentialist set theory in terms of it. Accepting this motivates favoring minimalist potentialism. For the dependence theorist can't take their core modal notion to be logical possibility because they want to say that certain logically contingent claims about what sets exist are necessary given the nature of sets[20].

Let me end this section by noting that I only claim to make a ceteris paribus case for minimalist formulations of potentialist set theory. One thing I could imagine shifting the scales in favor of dependence theory is a certain kind of

[18]Treating it as a logical operator requires taking its meaning must be held fixed when we're evaluating claims about logical possibility and entailment.

[19]We keep the idea that mathematical facts (at least those about set theory) are logical facts. But we reject all traditional logicist claims that mathematical objects (or anything else) exists as a matter of logic alone, conforming instead to mainstream mathematical practice of allowing that it would be logically possible for there to only be a single objects. We also wouldn't get any reason to think mathematical truths were cognitively trivial or knowable via linguistic competence alone, or knowable at all.

[20]For example it's logically contingent whether some set is an element of itself, but the dependence theorist will say that no such set could be generated.

progress regarding the liar paradox. It is generally agreed that paradoxes concerning the height of the hierarchy of sets and the liar paradox seem to have a lot in common, so it would be appealing to treat them similarly, if possible. Advocates of interpretational possibility based dependence set theory Linnebo and Studd see a close connection between their proposals and Kripke's approach to the Liar paradox (with determinate truth percolating up from some sentences to other sentences). So if Kripkian approaches to truth could be both vindicated and shown to have a uniquely close relationship to interpretational possibility or dependence theoretic approaches to set theory, this would improve the appeal of dependence potentialism. But, to my knowledge, no such results have been achieved.

4 Objections to Minimalist Set Theory

Let me end by discussing some objections to the currently most prominent form of minimalist set theory (Hellman's continuing development of potentialist set theory in works like [9, 10]). I will argue that these objections aren't a problem for minimalism itself, and can be avoided if we streamline/modify the basic concepts appealed to by Hellman in the way suggested in [2].

4.1 Metaphysical Shyness

First, I want to address an argument concerning the possibility of a kind of metaphysical or logical shyness which Linnebo gently raises for Hellman's minimalist potentialism in [13] a paper comparing the benefits of minimalist vs. dependence theoretic versions of potentailism.

In [13], Linnebo asks, "Do we really know that there cannot be 'metaphysically shy' objects, which can live comfortably in universes of small infinite cardinalities, but which would rather go out of existence than to cohabit with a larger infinite number of objects?" And he notes that the existence of such 'shy' objects would pose a problem for Hellman's minimalist potentialism, because it could block us from saying that every plurality of objects forming a hierarchy of a certain kind could be extended in a certain way.

In a similar vein, Linnebo notes that if Hellman's notion of logical possibility allows for an analog to metaphysically incompatible objects (e.g., two metaphysically possible knives formed by joining a single handle with different blades) this can make certain assumptions Hellman uses to justify the existence of potentialist translations ZFC come out false. In this way, Linnebo raises a very modest worry[21] about how/whether know that logical essences are suitably generous to satisfy the axioms Hellman accepts and uses to justify potentialist set theory. How do we know there couldn't be logically shy objects?

However, we should note that Linnebo himself doesn't make strong claims that this problem looks insolvable. Instead he notes that the minimalist can plausibly hope to solve it by understanding potentialist set theory as making claims about structure preserving extendability. For he writes "A ... promising option, suggested to me by Hellman, is to relax the extendability principle such that it only makes demands 'up to isomorphism': 'Necessarily, for any model M, possibly there is a model M0 which is isomorphic to M and which possibly has a proper extension.' While this is promising, we need to be shown how the modal structuralist has the resources to formulate the transworld isomorphism claim.". And [2] (which develops of potentialist set theory in terms of conditional/structure preserving logical possibility operator $\Diamond_{...}$) proposes one such formulation (though of course disputes about the right choice of logical primitives for metaphysics are always possible).

[21] His point in doing this is not to say that the latter claim can't be justified, but just to argue that understanding set theory as making claims about what's (logically) possible for all objects, rather than as making claims about what sets can be formed doesn't come without costs.

However this problem isn't very serious. Linnebo himself allows that Hellman would say that structure preservation is what matters, and Linnebo is sympathetic to this merely asking to see how this idea could be cashed out. And Berry[2] demonstrates one possible way of doing this cashing out [22].

I would actually suggest that considering our reactions to Linnebo's examples helpfully highlights a way that existing formulations of potentialism have fallen short of securing the full degree of structuralism that could be desired. For Linnebo's questions about shy objects highlight how typical formulations of potentialist set theory quantify in to the $\Diamond$ of logical/interpretational possibility forces us to consider when objects from one logically possible world are identical to or counterparts of objects in another. We are forced to ask whether, for some particular object, *that very object* could count as persisting in a world where the total universe has some cardinality, or some other possible object exists.

Yet (I think) intuitively such facts and claims about fragile essences should not matter to mathematics. Once we grasp and like the rough idea of potentialist set theory, what seems relevant to the truth of a '$\forall x \exists y$' claim is just whether the structure instantiated by some objects satisfying the width axioms could be preserved while these objects exist within a larger universe. It irrelevant to what we mathematically care about and what to say whether any of the objects we imagine forming this structure are shy (whether they'd persist through the introduction of additional objects or disappear as van Inwagen's in car does, when a car pulls out of the garage). Understanding minimalist potentialist set theory as making claims about structure-preserving extendability (what's pos-

[22] One might, of course, not like taking structure preservation as a primitive notion in this way: one might read Linnebo's remark more aggressively as demanding some way of understanding structure preserving logical possibility in terms of more familiar primatives. But insofar as the meaning of expressions that quantify in is vexed and mysterious and the meaning of structure preserving possibility is clear and agreed on, the methodological insistence that one must cash out structure preserving possibility in terms of claims about what's de re possible about some objects seems misguided or dogmatic. We owe no obligation to explain the lesser understood concept in terms of the better understood one.

sible given the structure of how 'set' and 'element' apply in a certain iterative hierarchy structure) fits into a long and celebrated tradition of seeing mathematics as somehow the science of structure. The right moral to draw isn't that the essences of sets is to be shy or gregarious or that we need to show that there can be no shy objects of any kind. Rather it's that we should avoid formulating mainstream mathematical claims so that their truth values depend on such vexed metaphysical questions about essence and persistence conditions.

4.2 Barcan-Marcus Problems?

The second concern I want to consider arises from controversies over quantified modal logic. A dependence potentialist might accept the motivations for formulating potentialist set theory in terms of logical possibility above, but ultimately reject this proposal for the following reason. We need to use quantified modal logic when formulating potentialist set theory (to talk about how a certain hierarchy of sets could be extended). And the right way to understand quantified modal logic must (contra Hellman and Kripke) make the converse Barcan-Marcus formula come out true. Accordingly, whatever modal notion we use must not allow the possibility of universes strictly smaller than our own. But (unlike dependence theorists Linnebo and Studd's notion of interpretational possibility) the intuitive notion of logical possibility above *does* allow such shrinking universes. For example, it's logically possible that there are only two objects). So we can't use logical possibility to formulate potentialist set theory.

Formulating potentialist set theory in terms of structure preservation (what is possible given the structural facts about how these relations apply?) rather than quantifying in (could *these very objects* exist within a certain kind of larger hierarchy?) solves this problem. For it ensures that reasoning about potentialist set theory won't require us to make any assumptions about objects having logical essences or that quantifying into the logical possibility operator is even well defined (much less accept any particular controversial axioms of quantified modal logic).

4.3 Inconsistencies and Justifying Replacement?

Finally, let me note that Roberts' inconsistency argument in [20] doesn't show the minimalist potentialist has a special problem justifying replacement. Arguably, both existing minimalist and dependence theoretic potentialists have a small problem justifying Replacement as fully as we might like. Major developments minimalist and dependence theoretic potentialism alike tend to just take their favored potentialist translation of replacement as an axiom, and admit that this is not particularly obvious seeming. For example, Hellman's book [9] and Linnebo's [14] take this line.

But Roberts' proof in [20] doesn't show there's any special further problem with Hellman's justification of Replacement. Rather, Roberts addresses a more ambitious project for justifying large cardinal axioms (not Replacement) which Hellman attempts in [20]. There Hellman points out that a certain very powerful and somewhat natural Reflection principle which would imply both all instances of Replacement and certain large cardinal axioms. Thus we might think that Hellman's potentialist Replacement axiom *abductively suggests* the relevant Reflection principle — which in turn implies large cardinal axioms. In [20] Roberts criticizes this project by showing that the relevant Reflection principle is inconsistent with principles that Hellman should accept. But the problem here concerns the *reflection* principle (and a certain way of motivating large cardinal axioms) not Hellman's potentialist *replacement* axiom itself.

5 Conclusion

In this paper, I have reviewed the motivations for adopting some potentialist approach to set theory and compared minimalist with dependence theoretic versions of potentialism. I have presented some arguments for favoring the minimalist version of potentialist set theory, and then responded to some crticisms and possible worries about this approach.

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